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INNER NEUMANN PROBLEM

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Abstract: This article discusses the main aspects of the formulation and solution of the Neumann problem for partial differential equations. The features of this boundary value problem, methods for its solution, as well as the necessary conditions for solvability are described. Particular attention is paid to the solution of the interior and exterior Neumann problems for a circular domain.

Keywords: Partial differential equations, Neumann problem, boundary conditions, method of separation of variables, Green's functions, interior and exterior Neumann problems, circular domain.

Introduction.

Partial differential equations are a powerful tool for modeling and describing a wide range of physics, engineering, and mathematics problems. One of the important classes of such equations is the Neumann problem, which considers the boundary conditions associated with the normal derivative of the solution. In this introduction we will consider the main aspects of the formulation and solution of the Neumann problem, as well as some of its generalizations and applications.

*1. Statement of the Neumann problem.*The Neumann problem is one of the fundamental boundary value problems for partial differential equations. Unlike the Dirichlet problem, where the value of the desired function is specified at the boundary of the domain, in the Neumann problem the value of the normal derivative of this function is specified at the boundary. Thus, the Neumann problem is formulated as follows: find a function u(x,y) that satisfies the differential equation in some domain Ω and the Neumann boundary condition on the boundary ∂Ω of this domain.

*2. Solution of the internal Neumann problem.*Various methods are used to solve the internal Neumann problem, including the separation of variables method, the integral equation method, and the Green's function method. These methods make it possible to obtain explicit expressions for solving the Neumann problem in various geometric configurations, such as a rectangle, circle, sphere, etc. Particularly important is the construction of Green's functions, which play a key role in representing the solution to the Neumann problem in the form of an integral equation.

*3. Internal Neumann problem in space.*The Neumann problem can be generalized to three-dimensional space, where instead of a flat region Ω, a three-dimensional region V is considered. In this case, the Neumann boundary condition is specified on the surface ∂V bounding the region V. Solving the internal Neumann problem in space requires the use of a more complex mathematical apparatus, including himself the theory of potentials, the theory of integral equations and methods of functional analysis. However, the basic principles for solving the internal Neumann problem in space are similar to the two-dimensional case.

*4. Necessary conditions for the solvability of the internal problem.*Neumann In order for the internal Neumann problem to have a solution, certain conditions must be met. These conditions are related to the consistency of the boundary data and the properties of the differential operator describing the equation in the domain. For example, for elliptic equations, a necessary condition is the fulfillment of the Neumann compatibility condition, which imposes restrictions on the boundary data. In addition, in the case of nonsmooth boundaries or nonsmooth coefficients of the equation, additional restrictions on the solvability of the Neumann problem may arise.

*5. Internal and external Neumann problems for a circle.*One of the important examples of solving the Neumann problem is the case of a circular region. For a circle, one can construct explicit solutions for both the internal and external Neumann problems. The solution to the internal Neumann problem for a circle is based on representing the solution in the form of a Fourier series in polar coordinates. Solving the external Neumann problem for a circle requires the use of potential theory and the construction of a fundamental solution to the Laplace equation outside the circle. These solutions are widely used in various fields such as electrostatics, fluid dynamics and elasticity theory.

Statement of the Neumann problem.

With regard to the type of areas bounded by surfaces, we will distinguish the following cases:(S)

A case that we will call ordinary. In this case, there is one surface bounding a solid body (Fig. 1).

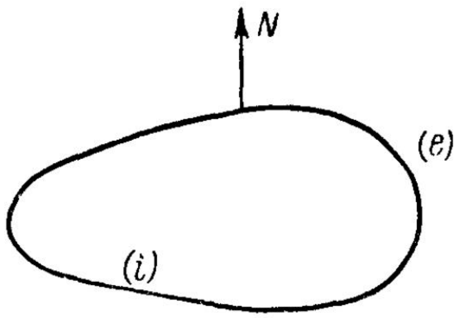

Figure 1. Surface bounding a solid body.

Of the two regions delimited by , we denote by the outer region containing the point at infinity, and by $(S)\left(D_e\right)\left(D_i\right)$ — inner area. The normal to the surface will always be directed to the area $N(S)\left(D_e\right)$.

The case in which, as shown in Fig. 2, there are several internal boundaries.

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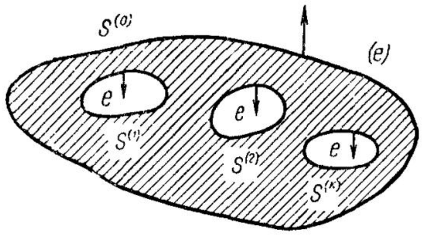
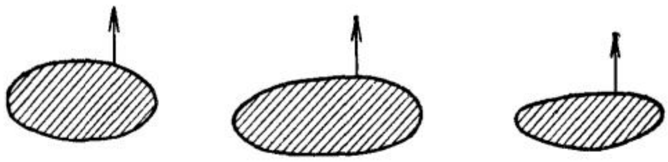


Figure 3. Single area, defined by all boundaries and containing point at infinity.



The normal is always considered to be directed at the set of boundaries; $N(D_e); S(S^{(1)}), (S^{(2)}), \dots, (S^{(k)})$ — boundaries of individual bodies. We will always assume that all boundaries satisfy the Lyapunov conditions.

Problem A. Find a function V that is harmonic inside the domain and satisfies the condition(D_i)

$$\frac{dV_i}{dn} = f \quad \text{in any point}(S), \qquad (1)$$

where f— given function, continuous on(S).

This is how the internal Neumann problem is formulated. Find a function that is internally harmonic and satisfies the condition$V(D_e)$

$$\frac{dV_e}{dn} = f \quad \text{in any point } (S). \qquad (2)$$

This is Neumann's external task.

It is easy to verify that in some cases for there exist nonzero solutions to the Neumann problem. Indeed, in the usual cases, a function equal to any constant in is harmonic and satisfies the condition on the boundary:$f = 0(J)U(D_i)(D_i)$

$$\frac{dU_i}{dn} = 0 \qquad (1')$$

In this case, the same condition is satisfied by a function that takes arbitrary constant values inside each of the surfaces. In the case of a function taking arbitrary constant values inside each of the surfaces and equal to zero outside, it is harmonic at and satisfies on the boundary the condition:$(E)U(S^{(1)}), \dots, (S^{(k)})(J)U(S^{(1)}), \dots, S^{(k)})(S^{(0)})(D_e)$

$$\frac{dU_e}{dn}=0. \qquad (2')$$

If a function that is harmonic inside [or in the case of] is a solution to the Neumann problem, then obviously it is a solution to the same Neumann problem. It can be shown that this limits the uncertainty in solving the Neumann problem. To do this, it is enough to show:$V(D_e)$ $V+U$

A function that is harmonic inside a finite connected region is equal to a constant if its normal derivative is zero at the boundary of the region;1°.

2°. A function that is harmonic inside an infinite connected domain with a finite boundary is equal to zero if its normal derivative is equal to zero.

Notes. For the case of , solving an external problem can be replaced by solving internal problems and one external one. in the same way, in the case of E, the solution of internal problems.k

Solution of the inner Neumann problem

An ordinary case.Let us note, first of all, that when following the internal Neumann problem, we cannot arbitrarily define the function Taking into account that the constant is a harmonic function in (D_i) , that f.

$$\int_S \frac{dV_i}{dn}ds=0 \qquad (3)$$

It follows that, given the condition

$$\frac{dV_i}{dn}=f \qquad (4)$$

equality (4) must also hold. So, condition (4) is a necessary condition for the possibility of this task. If the condition is met, the number cannot be a pole of a meromorphic function$\zeta=1$

$$\mu(0)=\frac{D_1(\zeta,0)}{D_1(\zeta)}=\rho_0+\zeta\rho_1+\zeta^2\rho_2+\ldots, \qquad (5)$$

in which

$$\rho_0=f,\ldots,\rho_n=-\frac{1}{2\pi}\int_{(S_1)}\rho_{n-1}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}d\sigma_1. \qquad (5')$$

A number cannot also be a pole of a function.$\zeta=1$

$$W=-\frac{1}{2\pi}\int_{(S_1)}\frac{\mu(1)d\sigma_1}{r_{10}}=V_1+\zeta V_2+\zeta^2V_3+\ldots,$$ Where

$$V_k = -\frac{1}{2\pi}\int_{(S_1)} \frac{\rho_k(1)d\sigma_1}{r_{10}} = -\frac{1}{2\pi}\int_{(S_1)} \frac{dV_{k-1}}{dn}\frac{d\sigma_1}{r_{10}}$$

Functions do not have poles, enclosed in the interval about t and cannot be their pole; therefore, we can substitute in the value We find $\zeta =-1 \zeta =+1; \zeta =-1 \zeta = 1$.

$$W = V_1 + V_2 + V_3 + \ldots, \qquad (4)$$

moreover,

$$\frac{dW_i}{dn} = -f.$$

So, the solution to the problem is determined by the following:

$$V = -V_1 - V_2 - V_3 - \ldots, \qquad (4' \qquad),$$

Based on the uniform convergence of the series

$$\mu_1(0) = \rho_0 - \rho_1 + \rho_2 + \ldots . (5)$$

on , we have: (S)

$$V = -\frac{1}{2\pi}\int_{(S_1)} \frac{\mu_1(0)d\sigma_1}{r_{10}} \qquad (6)$$

Thus, does have a normal derivative satisfying the condition V

$$\frac{dV_i}{dn} = f \quad (7)$$

for the function represents the solution of the equation at Note that the constant is a harmonic function in , the normal derivative of which is not zero; We can, therefore, when solving an internal problem, add a derivative constant to the found function. This remark is in complete agreement with the theorems proved above. For each solution of the equation at it is possible; this operation has the consequence of adding to some constant amount. Case . In the case of , as in the ordinary case, fulfillment is a necessary condition for the possibility of an internal task. The reasoning carried out above for the ordinary case remains valid for the granular case. But in case (J) the function can admit two poles: and When the condition is met, only the pole disappears. We cannot find the value of the function by substituting unlikely, since the radius of convergence of this series is equal to one. Multiplying the function by we get: $\zeta = 1.\ V(J).\ \ \zeta = 1 \zeta =-1;\ \ \zeta = 1.\ \ \zeta = W1 + \zeta;$

$$(1+\zeta)W = V_1 + (V_2 + V_1)^r + (V_3 + V_2)^{r^2} + \ldots + (V_n + V_{n-1})^{r^{n-1}} + \ldots (8)$$

Since a simple pole of a function, insofar as the value can no longer be a pole of a function, given that it is also not a pole of this function, we conclude that the radius of convergence of the last series is greater than one and that it is therefore possible to substitute a value into it We arrive at the formula$\zeta = 1$.

$$W = \frac{1}{2}\left| V_1 + (V_2 + V_1) + (V_3 + V_2) + \ldots + (V_n + V_{n-1}) + \ldots \right| \quad (9)$$

The required solution is obtained in the form of a series

$$W = -\frac{1}{2}\left| V_1 + (V_2 + V_1) + (V_3 + V_2) + \ldots + (V_n + V_{n-1}) + \ldots \right| \quad (10)$$

As in the ordinary case, it is possible to verify that the potential is indeed a simple layer and, therefore, admits a normal derivative that satisfies the conditions of the problem. The above remark regarding the most general solution to the problem remains valid: to obtain this solution, an arbitrary constant must be added to the found function. Case In case you can apply formula (18) to each area limited by the surface It follows that the internal Neumann problem is possible only if the conditions$V$. $(E)\left(S^{(7)}\right)$

$$\int_{(S^{(l)})} f d\sigma = 0 \qquad (l = 1, 2, \ldots, k). \quad (11)$$

If these conditions are satisfied, then cannot be a pole for functions (9) and (11). In the case under consideration, , for them there is also no pole, and they do not have poles and So, the radius of convergence of the series is greater than unity, the total in it , we have:$k\ \zeta = 1$

$$W = V_1 + V_2 + V_3 + \ldots, \quad (12)$$

The solution to the internal Neumann problem is obtained in the form of a series

$$V = -V_1 - V_2 - V_3 - \ldots, \quad (13)$$

We noticed that the study of this case could be replaced by a study of Neumann's internal problems. However, a function that would determine a solution for a region bounded by would only be relevant fork $\left(S^{(l)}\right)$ points inside this region, whereas the solution we just found gives the desired function in the form of one single series, suitable for all internal areas. To the solution found, I can add a function that remains constant in each of the areas bounded by surfaces, but whose value changes when moving from one area to another. This is in agreement with the lemma for the case when adding to the solution of a homogeneous equation, we only add constants that have the properties indicated above. Let us make a small remark about the connection between problems A and B. The condition is obtained as a necessary and sufficient condition for the solvability of problem B. Consequently, it is also a sufficient condition for the solvability of problem A. Is it necessary? Let us show that this is so. To be specific, let's consider an ordinary case. Let us assume that the harmonic function inside at the boundary satisfies the condition, and$V\left(S^{(l)}\right)U(D_i)(S)$

$$\int_{(S^{(l)})} f d\sigma = cS,\ c \quad 0. \qquad (14)$$

Then the function satisfies the condition , therefore, there is a potential of a simple layer such that Then the internally harmonic function $f_1 = f - cV \frac{dV_i}{dn} = f_1.(D_i)W = U - V$ at the boundary satisfies the condition(S)

$$\frac{dW_i}{dn} = c\ . \qquad (15)$$

If then this means that the function decreases if it moves along the normal from the boundary point and, therefore, the minimum point cannot be equal to the region, which contradicts the main property of the harmonic function. The same is impossible. This proves the necessity of condition (39) for the solvability of Problem A. It also follows that any solution to Problem A can be represented by the potential of a simple layer, and thus the equivalence of Problems A and B is proven.$c > 0\ \ W(D_i), c < 0$

<u>Internal Neumann problem in space.</u>

A function is called a solution to the internal Neumann problem if:$U(x, y, z)$

- $U \quad C^1(\bar{\Omega})$ (16)
- $U \quad C^2(\Omega)$
- $U = 0, (x, y, z) \quad \Omega$
- $\frac{\partial U}{\partial n}(x, y, z) = \psi(x, y, z)$
  (x, y, z)

The Theorem of uniqueness.

Let the functions $U_i(x, y, z)$, are such that:$i = 1,2$

- $U_i \quad C^1(\bar{\Omega})$ (17)
- $U_i \quad C^2(\Omega)$
- $U_i = 0, (x, y, z) \quad \Omega$
- $\frac{\partial U_i}{\partial n}(x, y, z) = \psi(x, y, z)$
  (x, y, z)

Then we get the following:

$$U_1(x, y, z) - U_2(x, y, z) = const$$

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Proof :

The difference in functions is seen:

$$U(x,y,z) = U_1(x,y,z) - U_2(x,y,z)$$

$$\bullet U \quad C^1(\bar{\Omega})$$

$$\bullet U \quad C^2(\Omega)$$

$$U = 0, (x,y,z) \quad \Omega$$

$$\frac{\partial U}{\partial n}(x,y,z) = \psi(x,y,z)$$

$$(x,y,z)$$

Green's first formula is used:

$$\underset{\Omega}{} U \quad v d\tau = \underset{\Sigma}{} U \frac{dv}{dn} d\sigma - \underset{\Omega}{} (grad\, U\ grad\, v) d\tau$$

$$\underset{\Omega}{} U \quad U d\tau = \underset{\Sigma}{} U \frac{dU}{dn} d\sigma - \underset{\Omega}{} (grad\, U\ grad\, U) d\tau \qquad (18)$$

$$\underset{\Omega}{} \frac{\partial U}{\partial x}^2 + \frac{\partial U}{\partial y}^2 + \frac{\partial U}{\partial z}^2 \quad d\tau = 0$$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = \frac{\partial U}{\partial z} = 0 \text{ в } \Omega$$

A uniqueness theorem for the interior Neumann problem was proved.

Necessary conditions for the solvability of the internal Neumann problem.

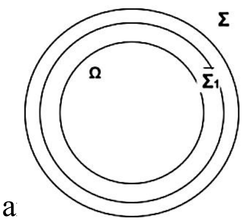

Figure 4. Area limited by surface Σ *and* $\bar{\Sigma}$

Region Ω limited by surface Σ .Function $U(x,y,z)$ harmonic in the a

$$\underset{\bar{\Sigma}}{} \frac{\partial U}{\partial n} d\sigma = 0 \qquad (19)$$

$$\underset{\Sigma}{} \frac{\partial U}{\partial n} d\sigma = 0, \bar{\Sigma} \to \Sigma$$

$\bar{\Sigma}$ -an arbitrary surface lying in the region Ω .

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$$\psi d\sigma_p = 0 \qquad (20)$$
$$\Sigma$$

Internal and external Neumann problems for a circle.

Obviously, in the case of a circle of radius with the center at the origin of the coordinates, the external normal derivative is$R$ $\left.\frac{\partial u}{\partial n}\right|_{\rho=R} = \left.\frac{\partial u}{\partial \rho}\right|_{\rho=R}$.

Therefore, the solution to the internal Neumann problem is sought in the form of a series

$$u(\rho,\varphi) = \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^{n} \left[a_n \cos(n\varphi) + b_n \cos(n\varphi)\right], \qquad (21)$$

where the coefficients n are determined from the boundary condition$a_n b_n$ $\left.\frac{\partial u}{\partial \rho}\right|_{\rho=R} = f(\varphi)$

$$a_n = \frac{R}{n\pi}\int_0^{2\pi} f(\varphi)\cos(n\varphi)d\varphi, b_n = \frac{R}{n\pi}\int_0^{2\pi} f(\varphi)\sin(n\varphi)d\varphi, \quad n = 1, 2, ...$$

The solution to the external Neumann problem is sought in the form of a series

$$u(\rho,\varphi) = \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^{-n} \left[a_n \cos(n\varphi) + b_n \cos(n\varphi)\right], \quad (22)$$

where the coefficients and .determined from the boundary conditiona_n b_n $\left.\frac{\partial u}{\partial \rho}\right|_{\rho=R} = f(\varphi)$ calculated using the same formulas $\left.\frac{\partial u}{\partial n}\right|_{\rho=R} = -\left.\frac{\partial u}{\partial \rho}\right|_{\rho=R}$.

Example 1. Find the steady-state temperature inside an unlimited cylinder of radius if a flow is given on its side surfaceRS $\left.\frac{\partial u}{\partial n}\right|_{S} = \cos^3\varphi$.

Solution. We need to solve the internal Neumann problem

$$u = 0, \qquad 0 < \rho < R, \quad 0 \quad \varphi < 2\pi,$$

$$\left.\frac{\partial u}{\partial \rho}\right|_{S} = \cos^3\varphi, \qquad 0 \quad \varphi \quad 2\pi,$$

First of all, it is necessary to check the fulfillment of the solvability condition for this Neumann problem, i.e. make sure that (Here the circumference of our circle).$C -$

In fact,

$$\int_C \frac{\partial u}{\partial n} ds = \int_0^{2\pi} \cos^3 \varphi R d\varphi = \frac{R}{2}\int_0^{2\pi} \cos\varphi d\varphi + \frac{R}{4}\int_0^{2\pi} \left[\cos(3\varphi) + \cos\varphi\right] d\varphi = 0. \quad (23)$$

Further, because $\cos^3 \varphi = \frac{3}{4}\cos\varphi + \frac{1}{4}\cos(3\varphi)$, then $a_1 = \frac{3}{4}R, a_3 = \frac{1}{12}R$ and that's all

the remaining coefficients in the series giving the solution to the internal problem

Neumann, go to zero. Therefore the solution has the form

$$u(\rho,\ \varphi) = C + \frac{3\rho}{4}\cos\varphi + \frac{\rho^3}{12R^2}\cos(3\varphi), \qquad (24)$$

where is an arbitrary constant. Comment. Neumann's problem can also be solved for a ring. The boundary conditions in this case will consist of specifying the external normal derivative:$C -$

$$-\frac{\partial u}{\partial \rho}(R_1, \varphi) = f_1(\varphi), \qquad \frac{\partial u}{\partial \rho}(R_2, \varphi) = f_2(\varphi). \qquad (25)$$

In this case, solving the problem is possible only if the condition is met

$$R_1 \int_0^{2\pi} f_1(\varphi) d\varphi = R_2 \int_0^{2\pi} f_2(\varphi) d\varphi \qquad (26)$$

and is determined up to an arbitrary constant.

Conclusion:

The Neumann problem is one of the fundamental boundary value problems of mathematical physics. It consists in finding a solution to a differential equation in a certain area, provided that at the boundary of this area the value of the derivative of the solution along the normal to the boundary is specified. This problem has wide application in various fields of science and technology, such as potential theory, elasticity theory, heat conduction theory and many others. The solution to the internal Neumann problem can be obtained using various methods, depending on the shape and size of the region, as well as on the boundary conditions. One of the most effective methods is the method of integral equations, which allows you to reduce the problem to solving an integral equation at the boundary of the region. The solution of this equation gives the value of the desired function on the boundary, and then, using Green's formula, one can find the solution inside the domain. The internal Neumann problem in space is to find a solution to a differential equation in a three-dimensional domain, provided that the value of the derivative is given at the boundary of this domain solutions normal to the boundary. This problem is more complex than the two-dimensional case and requires the use of more complex mathematical methods, such as potential theory and the theory of integral equations in space. In order for the internal Neumann problem to have a solution, a number of conditions must be met. Firstly, the region in which the problem is considered must be connected and have a sufficiently smooth boundary. Second, the boundary conditions must satisfy certain consistency conditions that ensure the existence of a solution. Thirdly, certain integral conditions must be satisfied at the boundary of the region, which relate the values of the solution's normal derivative with other characteristics of the problem. Consider an example of solving the internal and external Neumann problems for a circle. Let the area in which the problem is considered be a circle of radius R, the center of which coincides with the origin of coordinates. Neumann's internal problem is to find a solution to a differential equation in a circle, provided that the value of the normal derivative of the solution is specified on the boundary of the circle. The external Neumann problem consists of finding a solution to a differential equation outside a circle, provided that the normal derivative of the solution is given on the boundary of the circle. The solution to the internal Neumann problem for a circle can be obtained using the method of separation of variables. In this case, the solution is represented as a series, the coefficients of which are found from the boundary conditions. The solution to the external Neumann problem for a circle can be obtained using the potential method, in which the solution is represented in the form of a single or double layer potential distributed along the boundary of the circle.

Literature:

1. Korzyuk, V.I. Equations of mathematical physics: Textbook. Ed. 2nd, rev. and additional / V.I. Korzyuk. – M.: LENAND, 2021. – 480 p.

2. Erofeenko, V.T. Partial differential equations and mathematical models in economics: A course of lectures. Ed. stereotypical / V.T. Erofeenko, I. S. Kozlovskaya. – M.: Book house “LIBROKOM”, 2018. – 248 p.

3. Tikhonov, A.N. Equations of mathematical physics / A.N. Tikhonov, A. A. Samarsky. – M.: Nauka, 2017. – 736 p.

4. Apollonsky, S.M. Differential equations of mathematical physics in electronics. / CM. Apollonian. - St. Petersburg: Peter, 2012. - 352 p.

5. Gunther N.M. The theory was potential and its application to the main problems of mathematical physics. Edited by acad. Smirnova V.I. and prof. Smolitsky H.L. State edition technical and theoretical literature. Moscow 1953.-415 p.

6. Boss, V. Lectures on mathematics: Equations of mathematical physics / V. Boss. - M.: Lenand, 2018. - 224 p.

7.Vladimirov, V. Collection of problems on the equations of mathematical physics / V. Vladimirov. - M.: Fizmatlit, 2003. - 288 p.

8.Vladimirov, V.S. Collection of problems on equations of mathematical physics / V.S. Vladimirov, A.A. Vasharin et al. - M.: Fizmatlit, 2003. - 288 p.

9.Vladimirov, V.S. Collection of problems on equations of mathematical physics / V.S. Vladimirov, A.A. Vasharin et al. - M.: Fizmatlit, 2016. - 520 p.

10. Davydov, E.G. Equations of mathematical physics. Lecture notes / E.G. Davydov. - M.: Rusayns, 2019. - 16 p.

11. Derevich, I.V. Workshop on equations of mathematical physics: Textbook / I.V. Derevich. - St. Petersburg: Lan, 2018. - 428 p.

12. Dimitrienko, Yu.I. Differential equations of mathematical physics. Vol. XII / Yu.I. Dimitrienko. - M.: MSTU, 2011. - 367 p.

13.Emelyanov, V.M. Equations of mathematical physics. Workshop on problem solving: Textbook / V.M. Emelyanov, E.A. Rybakina. - St. Petersburg: Lan, 2008. - 224 p.

14.N. Teshavoeva. Mathematician physics methodology. Fergana. Ukituvchi. 1980.

15.M. Salokhiddinov. Mathematician physics tenglamalari. Tashkent. Uzbekistan. 2002.

16.M. T. Rabbimov. Mathematics. Tashkent. Fan ziyoshi. 2022. – 285 p.