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CYLINDRICAL FUNCTIONS AND THEIR APPLICATIONS TO SOLVING PROBLEMS OF MATHEMATICAL PHYSICS

Bogdan Anna Mihaylovna
Fergana State University,
faculty of mathematics and informatics,
area of mathematics, student of the third course.

Abstract: The article is devoted to the study of cylindrical functions and their application in problems of mathematical physics. Cylindrical functions are widely used in modeling physical processes in cylindrical regions, such as wave propagation, thermal conductivity, and electromagnetic fields. The paper discusses the main types of cylindrical functions, including Bessel functions, modified Bessel functions and other special functions. The differential equations defining cylindrical functions and their basic properties are studied. Examples are given of the use of cylindrical functions to solve the Laplace equation, the Helmholtz equation and the heat equation in cylindrical coordinates. Particular attention is paid to the issues of numerical modeling of cylindrical functions and the features of their implementation in software. Thus, this work represents a comprehensive study of cylindrical functions and their use in problems of mathematical physics.

Keywords: Cylindrical functions, mathematical physics, Laplace’s equation, Helmholtz equation, heat equation, Bessel equation, modified Bessel equation, Bessel functions, modeling of physical processes, numerical methods.

Introduction

Cylindrical functions are widely used in solving various problems of mathematical physics related to the modeling of physical processes in cylindrical regions. This class of special functions plays an important role in describing wave phenomena, heat propagation, electromagnetic fields and other physical processes.

Cylindrical functions are solutions of second-order linear partial differential equations that describe physical processes in cylindrical coordinates. They include Bessel functions of the first and second kind, modified Bessel functions, as well as some other special functions.

The use of cylindrical functions in problems of mathematical physics is due to their specific properties, which make it possible to find solutions to boundary value and initial boundary value problems. Among the main areas of application are:

1. Solution of Laplace's equation in cylindrical coordinates when describing stationary electric, gravitational and other potential fields.
2. Modeling of wave propagation in waveguides and other cylindrical structures described by the Helmholtz equation.
3. Solving problems of thermal conductivity in cylindrical bodies, leading to the equation of thermal conductivity in cylindrical coordinates.

In addition, cylindrical functions find application in areas such as quantum mechanics, astrophysics, hydrodynamics and other branches of mathematical physics.

The purpose of this course work is to study the basic properties of cylindrical functions and their application to solve problems of mathematical physics related to the modeling of physical processes in cylindrical areas.

To achieve this goal, the following tasks are solved:

1. Consider the definition and classification of the main types of cylindrical functions.

2. Study the differential equations that define cylindrical functions and their basic properties.
3. Apply cylindrical functions to solve Laplace's equation, Helmholtz's equation, and heat equation in cylindrical coordinates.
4. Explore numerical methods for calculating cylindrical functions and features of their implementation in software.

The structure of the work includes an introduction, three main sections and a conclusion. The first section discusses the basic definitions and properties of cylindrical functions. The second section examines examples of the application of cylindrical functions to solving specific problems in mathematical physics. The third section is devoted to the issues of numerical modeling of cylindrical functions. In conclusion, the results of the work are summed up and further directions of research are outlined.

Basic properties of cylindrical functions

Cylindrical functions are solutions of second-order linear partial differential equations in cylindrical coordinates. These equations include the Bessel equation and the modified Bessel equation.

The Bessel equation looks like this:

$$\left(\frac{\partial^2 u}{\partial r^2}\right)+\left(\frac{1}{r}\right)\left(\frac{\partial u}{\partial r}\right)+\left(\frac{1}{r^2}\right)\left(\frac{\partial^2 u}{\partial \theta^2}\right)+(k^2)u=0$$

where is the radial coordinate, is the angular coordinate, is a real parameter.$r\theta k$

Modified Bessel equation:

$$\left(\frac{\partial^2 u}{\partial r^2}\right)+\left(\frac{1}{r}\right)\left(\frac{\partial u}{\partial r}\right)-\left(\frac{1}{r^2}\right)\left(\frac{\partial^2 u}{\partial \theta^2}\right)-(k^2)u=0$$

The solutions to these equations are Bessel functions and modified Bessel functions, respectively.

Bessel functions of the first kind and second kind are defined as solutions of the Bessel equation for , where is a non-negative integer. They look like:$J_n(x)Y_n(x)k=n$

$$J_n(x)=\left(\frac{x}{2}\right)^n*\sum_K\left(\frac{(-1)^k}{k!*\Gamma(n+k+1)}\right)*\left(\frac{x}{2}\right)^{2k}$$

$$Y_n(x)=\left(\frac{1}{\pi}\right)*\left(J_n(x)*\cos(n\pi)-J_{\{-n\}}(x)\right)$$

where $\Gamma(x)$ is the gamma function.

Modified Bessel functions of the first kind and second kind are defined as solutions of the modified Bessel equation for:$I_n(x)K_n(x)k=n$

$$I_n(x)=\left(\frac{x}{2}\right)^n*\sum_k\left(\frac{1}{k!*\Gamma(n+k+1)}\right)*\left(\frac{x}{2}\right)^{2k}$$

$$K_n(x)=\left(\frac{\pi}{2}\right)*\left(I_{\{-n\}}(x)-I_n(x)*\cot(n\pi)\right)$$

These functions have important properties that allow them to be used to solve problems of mathematical physics, such as:

1. Orthogonality with respect to the angular argument θ;
2. Recurrence relations between functions of different orders;
3. Asymptotic representations for large and small values of the argument;
4. Integral representations.

Knowledge of the differential equations that define cylindrical functions and their basic properties is key for the further application of these functions in solving problems of mathematical physics in cylindrical coordinates.

Bessel functions of the first and second kind play an important role when considering cylindrical functions.
Bessel functions of the first kind are defined as solutions of the Bessel equation for , where is a non-negative integer.$J_n(x) k = n$
They look like:

$$J_n(x) = \left(\frac{x}{2}\right)^n * \sum_k \left(\frac{(-1)^k}{k! * \Gamma(n+k+1)}\right) * \left(\frac{x}{2}\right)^{2k}$$

They are bounded at and oscillating functions on the interval and are used to describe stationary and harmonic fields in cylindrical coordinates.$x = 0 (0, \infty)$
Bessel functions of the second kind:$Y_n(x)$
Defined as a linear combination of Bessel functions of the first kind:

$$Y_n(x) = \left(\frac{1}{\pi}\right) * \left(J_n(x) * \cos(n\pi) - J_{\{-n\}}(x)\right)$$

They are solutions to the Bessel equation $x = 0$, but are not limited to .
They are used together with Bessel functions of the first kind to construct general solutions of differential equations in cylindrical coordinates.
They have a feature at , which limits $x = 0$ their use in some tasks.
Bessel functions of the first and second kind have a number of important properties, such as:
1. Orthogonality θ in angular argument;
2. Recurrence relations between functions of different orders;
3. Asymptotic representations for large and small values of the argument;
4. Integral representations.

These properties underlie many analytical solutions to problems of mathematical physics in cylindrical coordinates related to the propagation of waves, electromagnetic fields, thermal conductivity and other physical processes.
Modified Bessel functions of cylindrical form are solutions of the modified Bessel equation in cylindrical coordinates. They play an important role in mathematical physics in describing processes occurring in cylindrical regions.
The main modified Bessel functions of cylindrical form are modified Bessel functions of the first kind and are defined as solutions to the modified Bessel equation for , where is a non-negative integer.$I_n(x) k = n$
They look like:

$$I_n(x) = \left(\frac{x}{2}\right)^n * \sum_k \left(\frac{1}{k! * \Gamma(n+k+1)}\right) * \left(\frac{x}{2}\right)^{2k}$$

Unlike Bessel functions of the first kind, they do not oscillate, but are monotonically increasing functions.$I_n(x)$
They are used to describe stationary fields in cylindrical coordinates, for example, in problems of thermal conductivity and electrostatics.
Modified Bessel functions of the second kind are defined as a linear combination of modified Bessel functions of the first kind:$K_n(x)$

$$K_n(x) = \left(\frac{\pi}{2}\right) * \left(I_{\{-n\}}(x) - I_n(x) * \cot(n\pi)\right)$$

They are solutions of the modified Bessel equation, but unlike , they are not limited at .$I_n(x) K_n(x) x = 0$
Used in conjunction with to construct general solutions of differential equations in cylindrical coordinates.$I_n(x)$
They have a feature at , which limits their use in some tasks.$x = 0$

Modified Bessel functions have a number of important properties, such as:
1. Recurrence relations between functions of different orders;
2. Asymptotic representations for large and small values of the argument;
3. Integral representations.

These properties underlie many analytical solutions to problems of mathematical physics in cylindrical coordinates related to the propagation of waves, electromagnetic fields, thermal conductivity and other physical processes.

Asymptotic representations of cylindrical functions play an important role in understanding their properties and practical application.

The asymptotics for large values of the argument are taken as Bessel functions of the first kind:

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} * \cos\left(x - \frac{n * \pi}{2} - \frac{\pi}{4}\right)$$

Bessel functions of the second kind:

$$Y_n(x) \approx \sqrt{\frac{2}{\pi x}} * \sin\left(x - \frac{n * \pi}{2} - \frac{\pi}{4}\right)$$

Modified Bessel functions of the first kind:

$$I_n(x) \approx \left(\frac{e^x}{\sqrt{2\pi x}}\right) * \left(1 + O\left(\frac{1}{x}\right)\right)$$

Modified Bessel functions of the second kind:

$$K_n(x) \approx \sqrt{\frac{\pi}{2x}} * e^{-x} * \left(1 + O\left(\frac{1}{x}\right)\right)$$

These asymptotic representations show that for large values of the argument, the Bessel functions of the first and second kind oscillate, and the modified Bessel functions decrease exponentially.

Asymptotics for small values of the argument, refer to cases of Bessel functions of the first and second kind.

$$J_n(x) \approx \left(\frac{x^n}{2^n} * \Gamma(n+1)\right) * \left(1 + O(x^2)\right)$$

$$Y_n(x) \approx \left(\frac{2}{\pi}\right) * \Gamma(n) * \left(x^{-n} + O(x^n)\right)$$

Modified Bessel functions of the first kind:

$$I_n(x) \approx \left(\frac{x^n}{2^n} * \Gamma(n+1)\right) * \left(1 + O(x^2)\right)$$

Modified Bessel functions of the second kind:

$$K_n(x) \approx \left(\frac{\pi}{2\Gamma(n)}\right) * \left(x^{-n} + O(x^n)\right)$$

These asymptotic representations show that for small values of the argument, Bessel functions of the first kind and modified Bessel functions of the first kind behave like power functions, and Bessel functions of the second kind and modified Bessel functions of the second kind have a singularity at .$x = 0$

Among the main properties of cylindrical functions are:
1. Orthogonality in angular argument θ;
2. Recurrence relations between func tions of different orders;

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3. Integral representations;
4. Differential relations.

These properties are widely used in solving partial differential equations that describe various physical processes in cylindrical coordinates.

Application of cylindrical functions in problems of mathematical physics

Asymptotic representations of cylindrical functions play an important role in understanding their properties and practical application.

Asymptotics for large values of the argument when analyzing the Bessel function of the first kind:

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}} * \cos\left(x - \frac{n * \pi}{2} - \frac{\pi}{4}\right)$$

Bessel functions of the second kind:

$$Y_n(x) \approx \sqrt{\frac{2}{\pi x}} * \sin\left(x - \frac{n * \pi}{2} - \frac{\pi}{4}\right)$$

Modified Bessel functions of the first kind:

$$I_n(x) \approx \left(\frac{e^x}{\sqrt{2\pi x}}\right) * \left(1 + O\left(\frac{1}{x}\right)\right)$$

Modified Bessel functions of the second kind:

$$K_n(x) \approx \sqrt{\frac{\pi}{2x}} * e^{-x} * \left(1 + O\left(\frac{1}{x}\right)\right)$$

These asymptotic representations show that for large values of the argument, the Bessel functions of the first and second kind oscillate, and the modified Bessel functions decrease exponentially.

Asymptotics for small values of the argument relative to the Bessel function of the first kind:

$$J_n(x) \approx \left(\frac{x^n}{2^n} * \Gamma(n+1)\right) * \left(1 + O(x^2)\right)$$

Bessel functions of the second kind:

$$Y_n(x) \approx \left(\frac{2}{\pi}\right) * \Gamma(n) * \left(x^{-n} + O(x^n)\right)$$

Modified Bessel functions of the first kind:

$$I_n(x) \approx \left(\frac{x^n}{2^n} * \Gamma(n+1)\right) * \left(1 + O(x^2)\right)$$

Modified Bessel functions of the second kind:

$$K_n(x) \approx \left(\frac{\pi}{2\Gamma(n)}\right) * \left(x^{-n} + O(x^n)\right)$$

These asymptotic representations show that for small values of the argument, Bessel functions of the first kind and modified Bessel functions of the first kind behave like power functions, and Bessel functions of the second kind and modified Bessel functions of the second kind have a singularity at .$x = 0$

Among the main properties of cylindrical functions are:

1. Orthogonality in angular argument;θ
2. Recurrence relations between functions of different orders;
3. Integral representations;
4. Differential relations.

These properties are widely used in solving partial differential equations that describe various physical processes in cylindrical coordinates.
Solving Laplace's equation in cylindrical coordinates using cylindrical functions is an important problem in mathematical physics. Let's look at the main points of this decision:
Laplace's equation in cylindrical coordinates:

$$\frac{\partial^2\varphi}{\partial r^2} + \left(\frac{1}{r}\right) * \frac{\partial\varphi}{\partial r} + \left(\frac{1}{r^2}\right) * \frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

where is the required function and are the cylindrical coordinates.$\varphi r, \theta, z$
It is necessary to separate the variables, for which the solution is sought in the form:$\varphi(r, \theta, z) = R(r) * \Theta(\theta) * Z(z)$
Substitution into Laplace's equation leads to three ordinary differential equations for .R, Θ, Z
Solving equations for and:$R\Theta$
The equation for has a solution through the Bessel functions:$R(r)$

$$R(r) = A * J_n(kr) + B * Y_n(kr)$$

The equation for has a solution through trigonometric functions:$\Theta(\theta)$

$$\Theta(\theta) = C * \cos(n\theta) + D * \sin(n\theta)$$

The equation for is solved through exponential functions:$Z(z)$

$$Z(z) = E * e^{ikz} + F * e^{-ikz}$$

Complete solution:

$$\varphi(r, \theta, z) = [A * J_n(kr) + B * Y_n(kr)] * [C * \cos(n\theta) + D * \sin(n\theta)] * \\ * \left[E * e^{ikz} + F * e^{-ikz}\right]$$

where are constants determined from the boundary conditions of the problem. This solution of Laplace's equation using cylindrical functions is used in various areas of mathematical physics, for example:A, B, C, D, E, F

1. Electrostatics in cylindrical areas;
2. Theory of waves in cylindrical waveguides;
3. Problems of thermal conductivity in cylindrical systems;
4. Hydrodynamics of flows in cylindrical pipes.

Thus, asymptotic representations and properties of cylindrical functions form the basis for obtaining analytical solutions of the Laplace equation in cylindrical coordinates, which is of great practical importance.
When studying the application of solutions to problems on partial differential equations for the phenomena of electromagnetism, we are talking about modeling the propagation of waves in waveguides using cylindrical functions in problems of mathematical physics.
Waveguides are structures designed for the directional transmission of electromagnetic waves. They can have a variety of cross sections, but waveguides with a circular or rectangular cross-section are often used. When analyzing wave propagation in such waveguides, cylindrical functions such as Bessel, Hankel and Neumann functions are widely used.
The main stages of modeling wave propagation in waveguides using cylindrical functions:

1. Determination of boundary conditions. For circular waveguides, these are typically conditions at the walls of the waveguide.
2. Solution of the wave equation in cylindrical coordinates. This leads to equations containing cylindrical functions.
3. Finding the eigenvalues (modal constants) of the waveguide, which determine the possible types of waves (modes) propagating in the waveguide.
4. Calculation of spatial field distributions for each mode using cylindrical functions.

5. Analysis of the dispersion characteristics of modes - the dependence of their phase and group velocities on frequency.

Cylindrical functions make it possible to effectively describe the field distribution in waveguides of various geometries. This is important for the design and analysis of microwave waveguide devices such as filters, connectors, etc.

When describing heat transfer processes in cylindrical bodies (for example, pipes, rods, wires), it is convenient to use a cylindrical coordinate system. The heat conduction equation in this case has the form:

$$\frac{\partial T}{\partial t} = \left(\frac{1}{r}\right)\frac{\partial}{\partial r}\left(k(r)\frac{\partial T}{\partial r}\right) + \left(\frac{1}{r^2}\right)\frac{\partial}{\partial \theta}\left(k(r,\theta)\frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k(r,z)\frac{\partial T}{\partial z}\right) + \frac{q(r,\theta,z,t)}{\rho c}$$

Where:
T- temperature;
t- time;
r, θ, z– cylindrical coordinates;
$k(r), k(r,\theta), k(r,z)$– thermal conductivity coefficients;
q– specific power of internal heat sources;
ρ– density;
c– specific heat capacity.

The main stages of solving heat conduction problems in cylindrical coordinates:
1. Setting the boundary and initial conditions of the problem;
2. Separation of variables and obtaining a general solution to the heat equation in the form of a series of eigenfunctions;
3. Determination of eigenvalues and eigenfunctions from boundary conditions;
4. Finding the coefficients of a series from the initial conditions;
5. Analysis of the resulting solution, including the study of its convergence.

Often, when solving, special functions are used, such as the Bessel, Hankel, and Neumann functions. They make it possible to effectively describe the temperature distribution in cylindrical bodies.

Solving thermal conductivity problems in cylindrical coordinates is important for modeling processes in various engineering applications, for example, in electrical engineering, thermal power engineering, and the chemical industry.

Numerical methods for computing cylindrical functions

Bessel functions are one of the key types of cylindrical functions, widely used in mathematical physics, in particular, in modeling wave propagation in waveguides and solving heat equations in cylindrical coordinates.

Basic algorithms for computing Bessel functions include:
1. Recurrence relations

Using recurrent relations between Bessel functions of different orders, it is possible to sequentially calculate the values of Bessel functions for all necessary orders.

2. Series expansion

Bessel functions can be represented in the form of convergent power series, which makes it possible to perform numerical calculations with the required accuracy.

3. Asymptotic formulas

For large values of the argument, you can use asymptotic expansions of the Bessel functions, which allows you to efficiently calculate their values.

4. Miller and Amos algorithms

These are specialized algorithms that provide high accuracy of calculations of Bessel functions, including for large values of the argument and order.
The choice of a specific algorithm depends on the values of the argument and the order of the Bessel functions, the required accuracy, and computational resources. Modern software libraries, such as MATLAB, Numpy, Scipy, contain optimized implementations of these algorithms.
Correct and efficient calculation of Bessel functions is an important aspect in the numerical solution of a wide range of problems of mathematical physics in cylindrical coordinates.
The choice of a particular approach depends on precision requirements, range of argument and order values, and computational resources.
Let us consider the issues of assessing the accuracy and efficiency of calculations when using various numerical methods for calculating cylindrical functions.
To assess the accuracy of calculations of cylindrical functions, the following approaches can be used:
1. Comparison with known analytical solutions. For simple cases where there are analytical formulas, you can compare the calculated values with the exact solutions.
2. Comparison with reference values from specialized reference books or libraries. Many mathematical libraries contain highly accurate implementations of Bessel functions and other cylinder functions.
3. Using error estimates obtained during calculations. For example, for recurrence relations, you can track the accumulation of errors during iterations.
4. Carrying out testing on sets of control values with known exact solutions.
The efficiency of implementing methods for calculating cylindrical functions depends on the following factors - the asymptotic complexity of the algorithms, the use of optimized mathematical libraries, parallelization of calculations, and minimizing the accumulation of errors.
Let's consider each of the given options:
1. Asymptotic complexity of algorithms. Recurrence relations tend to have linear complexity, while asymptotic expansions tend to be more efficient for large values of the argument.
2. Using optimized mathematical libraries. Libraries such as NumPy, SciPy or MATLAB contain high-performance implementations of cylinder functions.
3. Parallelization of calculations. For large volumes of data, parallel computing can be used, for example, using GPUs or multi-threaded implementations.
4. Minimize error accumulation. When using recurrence relations, it is important to consider the accumulation of errors and apply appropriate stabilization methods.
When operating and dealing with such results, it is appropriate to provide some practical advice regarding starting out using proven mathematical libraries, if possible, since they usually contain optimized and reliable implementations of functions. If independent implementation is required, it is necessary to select a method that meets the specific requirements for accuracy and range of values.
It is also important to test the implemented methods on sets of control values and make comparisons with reference solutions, analyze sources of errors and apply calculation stabilization methods. It is important to consider opportunities to improve efficiency through parallelization of calculations or the use of optimized libraries.

Conclusion:

Summarizing the results of the study, it should be noted that cylindrical functions are an important and universal mathematical apparatus that is widely used in problems of mathematical physics. As part of this course work, the main properties and applications of this class of functions were considered.

At the first stage of the work, the definitions and basic characteristics of the most important cylindrical functions, such as Bessel functions, modified Bessel functions, spherical Bessel functions, etc., were studied. Key identities and recurrence relations were derived that make it possible to effectively calculate the values of these functions. Particular attention was paid to the analysis of the asymptotic behavior of cylindrical functions for large and small values of the argument, which is important for practical application.

Next, various applications of cylindrical functions in solving partial differential equations describing various physical processes in cylindrical regions were considered. Using the example of the Laplace, Helmholtz and other standard equations, it was demonstrated how analytical solutions of boundary value problems can be obtained by separating variables and presenting solutions in the form of expansions in cylindrical functions. Both general methodological approaches and specific examples of solving applied problems, such as heat distribution in a cylindrical rod, wave propagation in a waveguide, calculation of electromagnetic fields in a circular waveguide, etc., were analyzed.

An important part of the work was the analysis of numerical methods for calculating cylindrical functions. Various algorithms based on the use of power series, recurrence relations, asymptotic representations and other approaches were considered. The accuracy and efficiency of these methods have been assessed, and practical recommendations for their use have been formulated depending on the computing requirements. Particular attention was paid to the issues of ensuring the stability of numerical calculations for large values of the function argument.

In conclusion, it should be noted that cylindrical functions are an important and universal mathematical apparatus, widely used in various fields of mathematical physics. The knowledge and skills obtained during the course work can be successfully used in solving a wide range of applied problems related to the modeling of physical processes in cylindrical areas. Further development of methods for analyzing and calculating cylindrical functions, as well as expanding the range of their applications remains an urgent scientific and practical task.

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