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INTEGRAL EQUATION OF HAMMERSTEIN'S TYPE WITH DEGENERATE KERNEL

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Abstract: In this paper the existence positive solution of the integral equation of Hammerstein type with degenerate kernel are discussed. We show how to find positive fixed points of a separable Hammerstein integral operator with a kernel to find positive solutions of a system of nonlinear algebraic equations in three unknowns.
Key words: Cone, continuous functions, integral equation, Hammerstein's operator, fixed point, system of nonlinear algebraic equations.

Let $C^+[0,1]$ is cone of space continuous functions on $[0,1]$. We define $C_0^+[0,1]=C^+[0,1]\setminus\{\theta\}$.
Consider the following functions $\varphi_1(x),\varphi_2(x),\varphi_3(x),\ \ \psi_1(x),\psi_2(x),\psi_3(x)\in C_0^+[0,1]$.
We define integral operator of Hammerstein type $H_k, k\in \mathbb{N}$ on the space C[0; 1]:

$$(H_k f)(t)=\int_0^1\left(\varphi_1(x)\psi_1(x)+\varphi_2(x)\psi_2(x)+\varphi_3(x)\psi_3(x)\right)f^k(u)du. \tag{1}$$

We study integral equation for fixed points of the Hammerstein's operator H_k:

$$H_k f=f,\quad f\in C_0^+[0,1], f(u)\geq 0 \tag{2}$$

We define positive numbers a_{ik} and b_{ik}:

$$a_{11}=\int_0^1\psi_1(u)\varphi_1^{\,2}(u)du \qquad a_{22}=\int_0^1\psi_1(u)\varphi_2^{\,2}(u)du$$

$$a_{33}=\int_0^1\psi_1(u)\varphi_3^{\,2}(u)du \qquad a_{12}=\int_0^1\psi_1(u)\varphi_1(u)\varphi_1(u)du$$

$$a_{13}=\int_0^1\psi_1(u)\varphi_1(u)\varphi_3(u)du \qquad a_{23}=\int_0^1\psi_1(u)\varphi_2(u)\varphi_3(u)du\ ;$$

$$b_{33}=\int_0^1\psi_2(u)\varphi_3^{\,2}(u)du. \qquad b_{12}=\int_0^1\psi_2(u)\varphi_1(u)\varphi_2(u)du$$

$$b_{13}=\int_0^1\psi_2(u)\varphi_1(u)\varphi_3(u)du \qquad b_{23}=\int_0^1\psi_2(u)\varphi_2(u)\varphi_3(u)du\ ;$$

$$c_{11}=\int_0^1 \psi_3(u)\varphi_1^{\,2}(u)du \qquad c_{22}=\int_0^1 \psi_3(u)\varphi_2^{\,2}(u)du$$

$$c_{33}=\int_0^1 \psi_3(u)\varphi_3^{\,2}(u)du \qquad c_{12}=\int_0^1 \psi_3(u)\varphi_1(u)\varphi_2(u)du$$

$$c_{13}=\int_0^1 \psi_3(u)\varphi_1(u)\varphi_3(u)du \qquad c_{23}=\int_0^1 \psi_3(u)\varphi_2(u)\varphi_3(u)du \quad ;$$

Consider operator $P_2:(x,y,z)\to(x',y',z')$ on the three dimensional space $\mathbb{R}^3$:

$$\begin{aligned}
x' &= a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{12}xy+2a_{13}xz+2a_{23}yz\\
y' &= b_{11}x^2+b_{22}y^2+b_{33}z^2+2b_{12}xy+2b_{13}xz+2b_{23}yz\\
z' &= c_{11}x^2+c_{22}y^2+c_{33}z^2+2c_{12}xy+2c_{13}xz+2c_{23}yz
\end{aligned} \qquad (3)$$

Lemma 1. Let be $k=2$. The Hammerstein's operator H_k has nontrivial positive fixed point iff the operator P_3 has nontrivial fixed point, moreover $N_{fix}^{+}(H_2)=N_{fix}^{+}(P_2)$. $N_{fix}^{+}(T)$ - T the number of nontrivial positive fixed points of the operator.

Isbot. (a) Let's enter the designations**:**

$$\mathbb{R}_3^{+}=\{(x,y,z)\in\mathbb{R}^3 : x\ge 0,\ y\ge 0\},$$
$$\mathbb{R}_3^{>}=\{(x,y,z)\in\mathbb{R}^3 : x> 0,\ y> 0\}.$$

Let $f(t)\in C_0^{+}[0,1]$ is nontrivial positive fixed point of the Hammerstein's operator H_k.We introduce the notations

$$s_1=\int_0^1 \psi_1(u)f^2(u)du \quad s_2=\int_0^1 \psi_2(u)f^2(u)du \quad s_3=\int_0^1 \psi_3(u)f^2(u)du \qquad (4)$$

From the equality $H_k f=f$ for fixed point f we have $f(t)=s_1\varphi_1(t)+s_2\varphi_2(t)+s_3\varphi_3(t)$. Clearly, that $s_1>0,\ s_2>0,\ s_3>0$ i.e. $(s_1,s_2,s_3)\in\mathbb{R}_3^{>}$. By the equalities (4) for parameters, s_1,s_2,s_3 we obtain the following equalities

$$\begin{aligned}
s_1 &= a_{11}s_1^2+a_{22}s_2^2+a_{33}s_3^2+2a_{12}s_1s_2+2a_{13}s_1s_3+2a_{23}s_2s_3\\
s_2 &= b_{11}s_1^2+b_{22}s_2^2+b_{33}s_3^2+2b_{12}s_1s_2+2b_{13}s_1s_3+2b_{23}s_2s_3\\
s_3 &= c_{11}s_1^2+c_{22}s_2^2+c_{33}s_3^2+2c_{12}s_1s_2+2c_{13}s_1s_3+2c_{23}s_2s_3
\end{aligned}$$

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It means the point (s_1, s_2, s_3) is fixed point of the operator P_k.

b) Let $\omega=(x_0,y_0,z_0)$ is nontrivial positive fixed point of the operator P_k, i.e. $\omega\in \mathbb{R}^3_+\setminus\{\theta\}$ and $P_k\omega=\omega$. Then

$$a_{11}x_0^2+a_{22}y_0^2+a_{33}z_0^2+2a_{12}x_0y_0+2a_{13}x_0z_0+2a_{23}y_0z_0=x_0$$
$$b_{11}x_0^2+b_{22}y_0^2+b_{33}z_0^2+2b_{12}x_0y_0+2b_{13}x_0z_0+2b_{23}y_0z_0=y_0$$
$$c_{11}x_0^2+c_{22}y_0^2+c_{33}z_0^2+2c_{12}x_0y_0+2c_{13}x_0z_0+2c_{23}y_0z_0=z_0$$

Using these equalities, we can verify that the function

$$f(t)=x_0\varphi_1(t)+y_0\varphi_2(t)+z_0\varphi_3(t)$$

is fixed point of the integral operator H_k.

We consider the following system of nonlinear algebraic equations with three unknowns:

$$\begin{cases} a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{12}xy+2a_{13}xz+2a_{23}yz=x \\ b_{11}x^2+b_{22}y^2+b_{33}z^2+2b_{12}xy+2b_{13}xz+2b_{23}yz=y \\ c_{11}x^2+c_{22}y^2+c_{33}z^2+2c_{12}xy+2c_{13}xz+2c_{23}yz=x \end{cases} \quad (5)$$

where $a_{ij}>0,\ b_{ij}>0,\ c_{ij}>0 \quad i,j=\{1,2,3\}$

Lemma 2. If $\omega=(x_0,y_0,z_0)\in R_+^3$ points is a positive solution of a system of nonlinear algebraic equations (5), then (u_0,v_0) points is a solution of the following system,

$$\begin{cases} \dfrac{a_{11}u^2+a_{22}t^2+a_{33}+2a_{12}ut+2a_{13}u+2a_{23}t}{c_{11}u^2+c_{22}t^2+c_{33}+2c_{12}ut+2c_{13}u+2c_{23}t}=u \\ \dfrac{b_{11}u^2+b_{22}t^2+b_{33}+2b_{12}ut+2b_{13}u+2b_{23}t}{c_{11}u^2+c_{22}t^2+c_{33}+2c_{12}ut+2c_{13}u+2c_{23}t}=t \end{cases} \quad (6)$$

where $u_0=\dfrac{x_0}{z_0},\quad t_0=\dfrac{y_0}{z_0}$

Lemma 3. If the point (u_0,t_0), $u_0>0, t_0>0$ is solution of a system (6) then the point $\omega_0=(u_0z_0,t_0z_0,z_0)\in R_+^3$ is a solution of the system of nonlinear algebraic equations (5), where

$$z_0 = \frac{1}{c_{11}u_0{}^2 + c_{22}t_0{}^2 + c_{33} + 2c_{12}u_0t_0 + 2c_{13}u_0 + 2c_{23}t_0}$$

We define the following polynomials:

$$g_1(t) = (2c_{11}b_{12} - a_{11}c_{11} - 2c_{12}b_{11})t + 2c_{11}b_{13} - 2c_{13}b_{11} + a_{11}b_{11};$$

$$g_2(t) = (b_{22}c_{11} - b_{11}c_{22} - 2a_{12}c_{11})t^2 + \\ +(2a_{11}b_{11} - 2b_{11}c_{23} + 2b_{23}c_{11} - 2a_{13}c_{11})t + 2a_{13}b_{11} - b_{11}c_{33} + b_{33}c_{11};$$

$$g_3(t) = a_{22}c_{11}t^3 + (2a_{23}c_{11} - a_{22}b_{11})t^2 + (a_{33}c_{11} - 2a_{23}b_{11})t - a_{33}b_{11};$$

$$h_3(t) = (2c_{11}^2a_{12} + c_{11}c_{22}b_{11} - c_{11}^2b_{22} + 4c_{12}c_{11}b_{12} - 2c_{12}a_{11}c_{11} - 4c_{12}^2b_{11})t^3 +$$
$$+\begin{pmatrix} 2c_{11}c_{23}b_{11} + 2c_{11}^2a_{13} - c_{22}b_{11}^2 - 2c_{11}^2b_{23} - 4b_{12}^2c_{11} + 4c_{13}c_{11}b_{12} + 2c_{12}a_{11}b_{11} - \\ -4c_{11}a_{12}b_{11} + 4c_{12}c_{11}b_{13} + 2a_{11}b_{12}c_{11} + 4b_{11}b_{12}c_{12} + b_{11}b_{22}c_{11} - 2a_{11}c_{11}c_{13} - 8b_{11}c_{12}c_{13} \end{pmatrix} t^2 +$$
$$+\begin{pmatrix} 2a_{12}b_{11}^2 + 2b_{11}b_{23}c_{11} + 2a_{11}b_{11}c_{13} + 2a_{11}b_{13}c_{11} + b_{11}c_{11}c_{33} + 4b_{11}b_{13}c_{12} + 4b_{13}c_{11}c_{13} + \\ +4b_{11}b_{13}c_{13} - b_{33}c_{11}^2 - 2b_{11}^2c_{23} - 2a_{11}b_{11}b_{12} - 4b_{11}c_{13}^2 - 4a_{13}b_{11}c_{11} - 8b_{12}b_{13}c_{11} \end{pmatrix} t +$$
$$+(b_{11}b_{33}c_{11} + 2a_{13}b_{11}^2 + 4b_{11}b_{13}c_{13} - 2a_{11}b_{11}b_{13} - 4c_{11}b_{13}^2),$$

$$h_4(t) = (2b_{12}c_{11}c_{12} - a_{11}c_{11}c_{22} - a_{22}c_{11}^2 - 2b_{11}c_{12})t^4 +$$
$$+\begin{pmatrix} a_{11}b_{22}c_{11} + a_{11}b_{11}c_{22} + 2a_{22}b_{11}c_{11} + 2b_{13}c_{11}c_{22} + 2b_{11}b_{22}c_{12} + 4b_{12}c_{11}c_{23} - \\ -2a_{11}c_{11}c_{23} - 2a_{23}c_{11}^2 - 2b_{11}c_{13}c_{22} - 2b_{12}b_{22}c_{11} - 4b_{11}c_{12}c_{23} \end{pmatrix} t^3 +$$
$$+\begin{pmatrix} 2a_{11}b_{23}c_{11} + 2a_{11}b_{11}c_{23} + 2b_{11}b_{22}c_{13} + 2b_{12}c_{11}c_{33} - a_{11}c_{11}c_{33} - a_{33}c_{11}^2 - \\ -a_{11}b_{11}b_{22} - a_{22}b_{11}^2 - 2b_{11}c_{12}c_{33} - 2b_{13}b_{22}c_{11} - 4b_{11}c_{13}c_{23} - 4b_{12}b_{23}c_{11} \end{pmatrix} t^2 +$$
$$+\begin{pmatrix} a_{11}b_{11}c_{33} + a_{11}b_{33}c_{11} + 2a_{33}b_{11}c_{11} + 2b_{11}b_{33}c_{12} + 2b_{13}c_{11}c_{33} + 4b_{11}b_{23}c_{23} - \\ -2a_{11}b_{11}b_{23} - 2a_{23}b_{11}^2 - 2b_{11}c_{13}c_{33} - 2b_{12}b_{33}c_{11} - 4b_{13}b_{23}c_{11} \end{pmatrix} t +$$
$$+(2b_{11}b_{33}c_{13} - a_{11}b_{11}b_{33} - a_{33}b_{11}^2 - 2b_{13}b_{33}c_{11}).$$

We define the following polynomial:

$$h_9(t) = (c_{11}t - b_{11})h_4^2(t) + 2(c_{12}t^2 + (c_{13} - b_{12})t - b_{13})h_4(t)h_3(t) + \\ +(c_{22}t^3 + (2c_{23} - b_{22})t^2 + (c_{33} - 2b_{23})t - b_{33})h_3(t) \qquad (6)$$

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Theorem.

(a) Suppose, for $t_0 \neq \frac{b_{11}}{c_{11}}$, $g_1(t_0) \neq 0$, $h_3(t_0) \neq 0$ conditions are appropriate and $t_0 \neq \frac{b_{11}}{c_{11}}$ number is positive solution $h_9(t) = 0$ equation. Let be $h_3(t_0) > 0,\ \ h_4(t_0) > 0$, then point (u_0, t_0) is solution of the system of nonlinear algebraic equations (6). Where

$$u_0 = \frac{h_4(t_0)}{h_3(t_0)};$$

(b) Suppose, for $t_1 = \frac{b_{11}}{c_{11}}$ $g_1(t_1) \neq 0$, $h_3(t_1) \neq 0$ conditions are appropriate . If u_1 is positive number, then point (u_1, t_1) is solution of the system of nonlinear algebraic equations (6). Where

$$u_1 = \frac{b_{11}^2 b_{22} c_{11} + b_{33} c_{11}^3 + 2 b_{11} b_{23} c_{23}^2 - b_{11}^3 c_{22} - b_{11} c_{11}^2 c_{33}}{2\left(b_{11}^2 c_{11} c_{12} + b_{11} c_{11}^2 c_{13} - b_{11} b_{12} c_{11}^2 - b_{13} c_{11}^2\right)}.$$

We define the following sets in $\mathbb{R}^3$:

$$D_1 = \{(x, y, z):\ b_{11} y - c_{11} z = 0,\ x > 0, y > 0,\ z > 0\},$$

$$D_2 = \{(x, y, z) \notin D_1:\ \tilde{b}_{11} y - \tilde{c}_{11} z = 0,\ x > 0, y > 0,\ z > 0\},$$

$$D_3 = \{(x, y, z) \notin D_2 \cup D_1 : \gamma_3 y^3 + \gamma_2 y^2 z + \gamma_1 y z^2 + \gamma_0 z^3 = 0,\ x > 0, y > 0,\ z > 0\},$$

$$D_4 = \{(x, y, z) \notin D_1 \cup D_2 \cup D_3,\ x > 0, y > 0, z > 0\},$$

where

$$\tilde{b}_{11} = 2 c_{11} b_{12} - a_{11} c_{11} - 2 c_{12} b_{11},\quad \tilde{c}_{11} = -2 c_{11} b_{13} + 2 c_{13} b_{11} - a_{11} b_{11},$$

$$\gamma_3 = 2 c_{11}^2 a_{12} + c_{11} c_{22} b_{11} - c_{11}^2 b_{22} + 4 c_{12} c_{11} b_{12} - 2 c_{12} a_{11} c_{11} - 4 c_{12}^2 b_{11},$$

$$\gamma_2 = 2 c_{11} c_{23} b_{11} + 2 c_{11}^2 a_{13} - c_{22} b_{11}^2 - 2 c_{11}^2 b_{23} - 4 b_{12}^2 c_{11} + 4 c_{13} c_{11} b_{12} + 2 c_{12} a_{11} b_{11} -$$
$$- 4 c_{11} a_{12} b_{11} + 4 c_{12} c_{11} b_{13} + 2 a_{11} b_{12} c_{11} + 4 b_{11} b_{12} c_{12} + b_{11} b_{22} c_{11} - 2 a_{11} c_{11} c_{13} - 8 b_{11} c_{12} c_{13},$$

$$\gamma_1 = 2a_{12}b_{11}^2 + 2b_{11}b_{23}c_{11} + 2a_{11}b_{11}c_{13} + 2a_{11}b_{13}c_{11} + b_{11}c_{11}c_{33} + 4b_{11}b_{13}c_{12} + 4b_{13}c_{11}c_{13} + $$
$$+ 4b_{11}b_{13}c_{13} - b_{33}c_{11}^2 - 2b_{11}^2c_{23} - 2a_{11}b_{11}b_{12} - 4b_{11}c_{13}^2 - 4a_{13}b_{11}c_{11} - 8b_{12}b_{13}c_{11},$$

$$\gamma_0 = b_{11}b_{33}c_{11} + 2a_{13}b_{11}^2 + 4b_{11}b_{13}c_{13} - 2a_{11}b_{11}b_{13} - 4c_{11}b_{13}^2.$$

Corollary

- **a)** the system of nonlinear algebraic equation (5) of number of positive solutions is at most one in sets D_1 and D_2 .
- **b)** the system of nonlinear algebraic equation (5) of number of positive solutions is no more than six in D_3 .
- **c)** the system of nonlinear algebraic equation (5) of number of positive solutions is no more than nine in D_4 .
- d) the system of nonlinear algebraic equation (5) of number of positive solutions is no more than seventeen.

REFERENCES

[1] Baxter, R.J. : Exactly Solved Models in Statistical Mechanics (Academic, London, 1982).

[2] Bleher, P.M., Ruiz, J. and Zagrebnov V.A. : On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice. Journ. Statist. Phys. 79 (1995), 473-482.

[3] Botirov G.I., A model with uncountable set of spin values on a Cayley tree: phase transitions// Positivity, 2016, DOI 10.1007/s11117-016-0445-x.

[4] Botirov G.I., Gibbs measures of a model on Cayley tree fixed points of the Hammerstein's operator //Conference series, 22-23 aprel 2016, 229-231.

[5] Eshkabilov Yu.Kh, Rozikov U.A., Botirov G.I.: Phase transition for a model with uncountable set of spin values

on Cayley tree. Lobachevskii Journal of Mathematics. (2013), V.34, No.3, 256-263.

[6] Eshkabilov Yu.Kh., Haydarov F.H., Rozikov U.A. : Uniqueness of Gibbs measure for models with uncountable

set of spin values on a Cayley tree. Math. Phys. Anal. Geom. 16(1) (2013), 1-17.

[7] Eshkabilov Yu.Kh., Haydarov F.H., Rozikov U.A. : Non-uniqueness of Gibbs measure for models with uncountable set of spin values on a Cayley Tree J.Stat.Phys. 147 (2012), 779-794.

[8] Eshkabilov Yu.Kh., Nodirov, Sh.D., Haydarov F.H.: Positive fixed points of quadratic operators and Gibbs Measures. Positivity. (2016), DOI: 10.1007/s11117-015-0394-9.

[9] Jahnel, B., Christof, K., Botirov, G. : Phase transition and critical value of nearest-neighbor system with uncountable local state space on Cayley tree. Math. Phys. Anal. Geom. 17 (2014) 323-331.

[10] Kotecky, R. and Shlosman, S.B. : First-order phase transition in large entropy lattice models. Commun. Math.Phys. 83 (1982), 493-515.

[11] Pigorov, S.A., Sinai, Ya.G. : Phase diagrams of classical lattice systems (Russian). Theor. and Math. Phys. 25(1975), 358-369.

[12] Pigorov, S.A., Sinai, Ya.G. : Phase diagrams of classical lattice systems. Continuation (Russian). Theor. andMath. Phys. 26 (1976), 61-76.

[13] Preston, C. : Gibbs states on countable sets (Cambridge University Press, London 1974).

[14] Rozikov, U.A. and Eshkabilov, Yu.Kh. : On models with uncountable set of spin values on a Cayley tree: Integral equations. Math. Phys. Anal. Geom. 13 (2010), 275-286.