INTERNATIONAL MULTIDISCIPLINARY JOURNAL FOR RESEARCH & DEVELOPMENT
SJIF 2019: 5.222 2020: 5.552 2021: 5.637 2022:5.479 2023:6.563 2024: 7,805
eISSN :2394-6334 https://www.ijmrd.in/index.php/imjrd Volume 12, issue 07 (2025)

AREA OF SEPARATE-HARMONICITY

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Abstract: The paper considers the concept of separate harmonicity of functions of several variables with respect to disjoint groups of variables. The separate harmonicity hulls for domains are studied. In particular, it is proven that a domain possessing barrier functions on a dense subset of the boundary is a domain of separate harmonicity.

Keywords: Harmonic function, separately harmonic function, separate harmonicity hull, barrier, domain of separate harmonicity

Аннотация: В работе рассматривается понятие сепаратно-гармоничности функций многих переменных по раздельным группам переменных. Исследуются оболочки сепаратно-гармоничности для областей. В частности, доказано, что область обладающего барьерными функциями на всюду плотном подмножестве границы, является областью сепаратно-гармоничности.
Ключевые слова: Гармоническая функция, сепаратно-гармоническая функция, оболочка сепаратно-гармоничности, барьер, область сепаратно-гармоничности.

1.Introduction

It is known that for any flat domain in D there exists a function holomorphic in D and not analytically continued beyond this domain, i.e. every flat domain is a domain of holomorphy. In contrast, in the space ^{n}, $n>1$, there exist domains from which any holomorphic function can be analytically continued into a wider domain. For example, a non-simply connected domain $\left\{z \quad ^n : 1<|z|<2\right\}$ is an example of such domains (see [1, p.148, Osgood-Brown theorem]). This means that in ^{n}, $n>1$, every domain is not a domain of holomorphy, i.e. the class holomorphic functions of several complex variables have the effect of forced analytic continuation (see [1, p.153]).

It turns out that the class of separately harmonic functions also has the property of forced analytic continuation. In this paper, we study the domains of separately harmonicity.

2. Separate harmonic functions

Let $x \quad \mathrm{R}^n$, $y \quad \mathrm{R}^m$, $\quad \mathrm{R}^n \quad \mathrm{R}^m \quad D \quad G$ — be a domain, $D \quad E$ and $G \quad F$ — some subsets. Suppose that, following the function $u(x,y)$, initially defined on the set $E \quad F$, has the properties:

a) for any fixed $x^0 \quad E$, the function $u(x^0,y)$ continues harmonically in G;

b) for any fixed $y^0 \quad F$ the function $u(x,y^0)$ continues harmonically in D.

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In this case, the specified extensions and $u(x,y)$ define a certain function on the set $X=(E\times G)\cup(D\times F)$, which is called a separate harmonic function on X.
In the case when in $E=D$ and $F=G$, the function $u(x,y)$ is called separately harmonic in the region $X=D\times G$, i.e. harmonic with respect to the groups of variables separately.
The set X is not domain in general. For an arbitrary domain that cannot be represented as a product of two domains, the separately harmonic function is defined as follows: if the function $u(x,y)$ is defined in the domain $\Omega\subset \mathbb{R}^n(x)\times\mathbb{R}^m(y)$, $n,m>1$, and has the following properties.

1) for any $x^0:\{x=x^0\}\cap\Omega\neq\emptyset$, function $u(x^0,y)$ harmonic with respect to variable y on the section $\{x=x^0\}\cap\Omega$;

2) for any $y^0:\{y=y^0\}\cap\Omega\neq\emptyset$, the function $u(x,y^0)$ is harmonic in the variable x on the section $\{y=y^0\}\cap\Omega$, then it is called a separately harmonic function in region Ω.

The above defined class of separately harmonic functions is denoted by $h_{nm}(\Omega)$ $(h_{nm}(X))$.
The well-known Hartogs theorem (see **[1]**) states that if a function $f(z,w)$ is holomorphic in the domain $D\subset\mathbb{C}^n$ for fixed w and it is holomorphic in $G\subset\mathbb{C}^m$ holomorphic in the domain $D\times G\subset\mathbb{C}^n(z)\times\mathbb{C}^m(w)$ with respect to the set of variables.
In 1961, P. Lelong [2] proved the following analogue of this theorem for separately harmonic functions: if the function $u(x,y)$ is separately harmonic in the domain $D\times G\subset\mathbb{R}^n(x)\times\mathbb{R}^m(y)$, then the function $u(x,y)$ is harmonic in $D\times G$ with respect to the set of variables.
Let us now consider the following general problem: let in $E\subset D\subset\mathbb{R}^n$, $F\subset G\subset\mathbb{R}^m$ and $u(x,y)$ separately harmonic in the set it is required to describe the $X=(E\times G)\cup(D\times F)$ the harmonic domain functions $u(x,y)$ be .
This problem was studied in the works **[3]**, **[4]**), **[6]**, [8], [9].
In 1982, A. Zeryakhi [4] obtained the following result: let the domain of satisfying $D\times G$ — space $\mathbb{R}^2(x)\times\mathbb{R}^2(y)$ and $E\subset D$, $F\subset G$ — be compact sets the conditions of H regularity in classes of harmonic polynomials. Then any separately harmonic function on the set $X=(E\times G)\cup(D\times F)$ is harmonically continued into the domain

$$X=\{(x,y)\in D\times G:\omega^*_{sh}(x,E,D)+\omega^*_{sh}(y,F,G)<1\}.$$

Usually, to continue harmonic functions, one first goes to holomorphic functions and then uses the principles of holomorphic continuations.
The following result plays a fundamental role in the study of continuation of harmonic functions.

Preposition 1 (see [6]). Consider the space $\mathbb{R}^n(x)$, embedded in $\mathbb{C}^n(z)=\mathbb{R}^n(x)+i\,\mathbb{R}^n(y)$, where $z=(z_1,\ldots,z_n)$, $z_j=x_j+i\,y_j$, $j=1,\ldots,n$, and let

D be some bounded domain from $R^n(x)$. Then there exists a domain $D \quad C^n(z)$ such that in $D \quad D$ and for any function $u(x) \quad h(D)$ there exists a holomorphic function $f_u(z)$ in D such that $f_u|_D = u$ is a subdomain. In addition, for any number $M > 1$ there exists $D_M \quad D$, $D \quad D_M$, such that $\|f_u\|_{D_M} \quad M\|u\|_D$, $\forall u \quad h(D) \cap L \quad (D)$ (here $\|u\|_D = \sup\{|u(x)| : x \quad D\}$).

Теорема 1 ([6]). Let $E \quad D \quad R^n$ and $F \quad G \quad R^m$ be sets that are not pluripolar compact in the sense of subsets of the spaces $C^n(z) = R^n(x) + i \ R^n(y)$ and $C^m = R^m + i \ R^m$. Any separately harmonic function $u(x,y)$ on the set $X = (E \quad G) \cup (D \quad F)$ can be harmonically extended to the domain

$$X = \{(x,y) \quad D \quad G : \omega^*(x,E,D) + \omega^*(y,F,G) < 1\}.$$

Thus, if the compact sets $E = \{x \quad R^n : |x| \quad R_1\}$, $F = \{x \quad R^m : |x| \quad R_2\}$), then the function $u(x,y)$ is continued into some neighborhood of $E \quad F$.

Here

$$\omega(z,E,D) = \sup\{u(z) : u \quad psh(D), u|_E \quad 0, u|_D \quad 1\}.$$

$$\omega^*(z,E,D) = \overline{\lim_{z \to z}} \omega(z,E,D), \ z \quad D,$$

is called the P-measure (plurisubharmonic measure) of the set $E \quad D$ with respect to the domain in $D \quad {}^n$ (see [5], [7]).

3. Main theorems

From the above theorems it is easy to verify that for some domains with $\Omega \quad {}^{n+m}$ there exists a corresponding, wider domain of harmonic $\hat{\Omega} \quad \Omega$ such that the class of separately functions domain is called the envelope of separately-harmonicty to Ω.

Definition 1. The $\Omega \quad {}^{n+m}$ domain is called a domain of separate harmonicity for class $h_{nm}(\Omega)$, if there exists some function $u(x,y) \quad h_{nm}(\Omega)$ that does not continue at any point outside Ω which is harmonically.

Definition 2. We will say that at the boundary point $(\xi,\eta) \quad \partial\Omega$ to the domain $\Omega \quad {}^{n+m}$ there is a barriers if there exists a function $\vartheta(x,y) \quad h_{nm}(\Omega)$, unbounded at the point (ξ,η), i.e. $\vartheta(x_j,y_j) \to \quad$ of the sequence $(x_j,y_j) \quad \Omega$, $(x_j,y_j) \to (\xi,\eta)$ at $j \to + \quad$.

Theorem 2. If on an everywhere dense set of points of the boundary of a domain Ω there exists a barrier from class $h_{nm}(\Omega)$, then Ω is a domain of separate harmonicity for class $h_{nm}(\Omega)$.

Proof. Let the domain $\Omega \quad {}^{n+m}$ have an everywhere dense countable set of barrier points $\{(\xi_1,\eta_1),(\xi_2,\eta_2),\ldots,(\xi_j,\eta_j),\ldots\} \quad \partial\Omega$. Then, by definition, there exists a sequence of functions

$\vartheta_j(x,y) \in h_{nm}(\Omega)$, $\overline{\lim\limits_{(x,y)\to(\xi_j,\eta_j)}} \vartheta_j(x,y) = +\infty$. We take a sequence of domains Ω_j : $\Omega_j \Subset \Omega_{j+1} \Subset \Omega$, $\bigcup\limits_{j=1}^{+\infty} \Omega_j = \Omega$ and consider the sum of the following series

$$\vartheta(x,y) = \sum_{j=1}^{+\infty} \frac{1}{j^2 c_j} \vartheta_j(x,y), \quad c_j = \max\left\{\vartheta_j(x,y) : (x,y) \in \Omega_j\right\}.$$

It is clear that the function $\vartheta(x,y)$ belongs to the class $h_{nm}(\Omega)$ is not harmonically continued at any point outside Ω. **The theorem 2 is proved.**

Let D_n and D_m be domains from $\mathbb{R}^n$ and $\mathbb{R}^m$, $D_n \supset O_n$, $D_m \supset O_m$ respectively, an opening subset,and $X_{nm} = (O_n \times D_m) \bigcup (D_n \times O_m)$ an open set of cross type. Denote by $\widehat{X}_{nm}$ the maximal open set such that $X_{nm} \subset \widehat{X}_{nm}$, $h_{nm}(\widehat{X}_{nm}) = h_{nm}(X_{nm})$ (see Theorem 1).

Theorem 3. The domain $\Omega \subset \mathbb{R}^{n+m}$ is a domain of separate harmonicity if and only if for any open set of X_{nm} type crosses belonging to Ω, its separate harmonic hulls also belong to Ω: $X_{nm} \subset \Omega \Rightarrow \widehat{X}_{nm} \subset \Omega$.

Corollary 1. Any domain of type $D_n \times D_m \subset \mathbb{R}^{n+m}$ is domain of separate harmonicity for the class $h_{nm}(D_n \times D_m)$.

Corrolary 2. Any convex domain $\Omega \subset \mathbb{R}^{n+m}$ is a domain of separate harmonicity for class $h_{nm}(\Omega)$.

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