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MATHEMATICAL EXPECTATIONS OF FUNCTIONS FROM RANDOM SUMS

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1. Introduction

A random sum is the sum of a random number of random variables. Instead of adding a fixed number of terms, the number of terms itself is uncertain. This structure appears naturally in many applied probability models.

The main problem studied in this topic is how to compute or analyze the expected value of a function of such a random sum. This is more complicated than finding the expectation of a simple random variable because two layers of randomness are involved:

- randomness in the number of terms,
- randomness in the terms themselves.

Let a sequence of random variables be defined on some probability space (Ω, F, P)

$$\xi_1, \xi_2, \ldots, \xi_n, \ldots \tag{1}$$

and a non-negative integer random variable ν.
In the theory of summation of random variables, a random sum is a quantity

$$S_\nu = \xi_1 + \xi_2 + \ldots + \xi_\nu,$$

i.e. the sum of a random number of random variables (see [4]). In the case where the sequence (1) consists of non-negative independent random variables, the random sum S_ν plays a significant role in the theory of branching random processes ([5], [8]), as well as in reliability theory ([1]). For example, for the study of the asymptotic properties of branching random processes (the probabilities of degeneration and continuation of the process, the limit distributions of the sum of the total number of particles, etc.), B.A. Sevastyanov's problem of describing the class of measurable functions $g(x)$ defined on $[0, \ )$ such that from the existence of $Eg(\xi_i)$ $(i \quad \mathbf{N})$ and $Eg(\nu)$ the existence of mathematical expectation follows $Eg(S_\nu)$.

We first examine the formulated problem for random sums S_ν, where the terms ξ_1, ξ_2,... are independent random variables and ν is an independent integer non-negative random variable. According to the formula for the total mathematical expectation, the characteristic function of a random sum S_ν has the following form:

$$\Psi(t) = Ee^{itS_\nu} = \sum_{n=0} E(e^{itS_n}; \nu = n) = \sum_{n=0} P(\nu = n) Ee^{itS_n} =$$

$$= \sum_{n=0} P(\nu = n) \prod_{k=1}^{n} f_k(t), \tag{2}$$

where $S_0 = 0$, $f_k(t) = Ee^{it\xi_k}$ is the characteristic function of random variables. ξ_k. From here on, the entry $E(\xi; A)$ means that the averaging of random variables ξ is performed over a set A, , i.e.

$$E(\xi; A) = \int_A \xi(\omega) P(d\omega), \quad A \quad F.$$

Taking into account the formulas $ES_\nu^r = i^{-r}\Psi^{(r)}(0)$ from equality (2), we conclude that the existence of a mathematical expectation $max(E\nu^r, E\xi_i^r) <$, i $\mathbf{N}$, for an integer , $r > 0$ entails $ES_\nu^r <$, and the value ES_ν^r is a function of the values $E\xi_i^{r_1}$, $E\nu^{r_1}$ for integers r_1 $(0 < r_1 \quad r)$. For example, in the case when random variables ξ_i are identically distributed (see [4]),

$$ES_\nu = E\nu E\xi_1, \; ES_\nu^2 = E\nu[E\xi_1^2 - (E\xi_1)^2] + E\nu^2(E\xi_1)^2$$

etc.

From the above it follows that bounded functions and power functions $g(x) = x^r$, $r = 1, 2, ...$ belong to the class of measurable functions possessing the properties formulated in the given problem of B.A. Sevastyanov . Below we will establish fairly broad conditions for functions to belong $g(\;)$ to the class of functions considered in this problem.

Following B.A. Sevastyanov [5], we introduce the following classes of functions:

1) We will say that a function $g(x)$ defined on $[0, \;)$ belongs to the class G_1 if $g(x)$ it is non-negative and there exists such $C > 0$ that for any $x, y \quad 0$

$$g(xy) \quad Cg(x)g(y). \tag{3}$$

2) We will say that a function $g(x)$ defined on $[0, \;)$ belongs to the class G_2 if it is non-negative, non-decreasing and convex.

The following theorem holds:

Theorem 1 [5]. Let the random variables ξ_i of sequence (1) be independent and identically distributed, and let be ν an independent integer non-negative random variable. Then for any function $g \quad G_1 \quad G_2$ the inequality holds.

$$Eg(S_\nu) \quad CEg(\xi_1)Eg(\nu). \tag{4}$$

The monograph [5] presents a proof of Theorem 1 and an application of inequality (4) to the theory of branching random processes. Below we present another proof of this theorem, based on the formula for complete mathematical expectations.

Proof of Theorem 1. Due to convexity $g(x)$ $(g \quad G_2)$ and from inequality (3) $(g \quad G_1)$ with probability 1 we have

$$g(S_\nu) = g(\xi_1 + \xi_2 + \ldots + \xi_\nu) = g \quad \frac{\xi_1\nu + \xi_2\nu + \ldots + \xi_\nu\nu}{\nu}$$

$$\frac{1}{\nu} \quad {}_{i=1}^{\nu} g(\xi_i\nu) \quad C\frac{g(\nu)}{\nu} \quad {}_{i=1}^{\nu} g(\xi_i). \tag{5}$$

Further, according to the formula for the total mathematical expectation and by virtue of inequality (5)

$$E\left[g(S_\nu)\right] = {}_{n=1} E\left[g(S_\nu);\, \nu = n\right] = {}_{n=1} E\left[g(S_n);\, \nu = n\right]$$

$$C \quad {}_{n=1} \frac{g(n)}{n} \quad {}_{k=1}^{n} E[g(\xi_k);\, \nu = n] \quad . \tag{6}$$

Considering the identical distribution of random variables ξ_i and their independence from ν, we have

$$E[g(\xi_k); \nu = n] = P(\nu = n)Eg(\xi_1). \tag{7}$$

From relations (6), (7) we finally obtain

$$E\left[g(S_\nu)\right] \quad CE\left[g(\xi_1)\right] \quad {}_{n=1} g(n)P(\nu = n) = CE\left[g(\xi_1)\right]E\left[g(\nu)\right].$$

The proof of Theorem 1 is complete.

Remark 1. As shown by B.A. Sevastyanov [5], the conditions for the existence $Eg(\nu) <$ and membership of a function g in the class of convex functions G_2 cannot be weakened.

Remarks 2. Functions from the class G_1 with a constant $C = 1$ are called semi-multiplicative. From the proof of Theorem 1, we can conclude that if the convexity conditions of functions $g()$ are replaced by the concavity condition, then lower bounds for can be obtained $Eg(S_\nu)$. It should also be noted that semi-multiplicative convex functions play an important role in the theory of linear operators (see [2], [3]).

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