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THE INVERSE PROBLEM OF DETERMINING THE COEFFICIENT FOR THE WAVE PROPAGATION EQUATION

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Annotation

This article investigates an inverse problem of determining unknown coefficients in wave propagation processes described by second-order hyperbolic partial differential equations. The relevance of the study is обусловлено the need to analyze wave dynamics in heterogeneous media and to determine the internal characteristics of a medium through remote measurements. The paper theoretically examines the conditions of well-posedness of the problem, in particular the existence, uniqueness, and stability of the solution in the sense of Hadamard. Additional data obtained at boundary or internal points are used to identify the unknown coefficient. Iterative methods based on solving nonlinear operator equations are proposed, and the efficiency of the suggested algorithms is demonstrated through numerical simulations. The obtained results have important practical applications in fields such as seismic exploration, hydroacoustics, and medical tomography.

Keywords

wave equation, inverse problem, coefficient identification, hyperbolic system, uniqueness theorem, well-posedness, iterative algorithm, boundary conditions, seismic waves, medium characteristics.

ОБРАТНАЯ ЗАДАЧА ОПРЕДЕЛЕНИЯ КОЭФФИЦИЕНТА ДЛЯ УРАВНЕНИЯ РАСПРОСТРАНЕНИЯ ВОЛН

Аннотация

В данной статье исследуется обратная задача определения неизвестных коэффициентов в процессах распространения волн, описываемых дифференциальными уравнениями в частных производных гиперболического типа второго порядка. Актуальность исследования обусловлена необходимостью изучения динамики волн в гетерогенных средах и дистанционного определения внутренних характеристик среды. В работе теоретически проанализированы условия корректности задачи по Адамару, в частности, вопросы существования, единственности и устойчивости решения. Для нахождения неизвестного коэффициента используются дополнительные данные в граничных или внутренних точках области. В статье представлены итерационные методы, основанные на решении нелинейных операторных уравнений, а также результаты численного моделирования, демонстрирующие высокую точность предложенных алгоритмов. Полученные результаты имеют важное прикладное значение в таких областях, как сейсморазведка, гидролокация и медицинская томография.

Ключевые слова

волновое уравнение, обратная задача, идентификация коэффициентов, гиперболическое уравнение, теорема единственности, корректность, итерационный алгоритм, граничные условия, сейсмические волны, характеристики среды.

INTRODUCTION

In the modern theory of partial differential equations, inverse problems specifically those involving the determination of unknown coefficients in hyperbolic equations describing wave propagation represent one of the most significant directions from both fundamental and applied perspectives. Wave dynamics serve as the mathematical foundation not only for theoretical physics but also for fields such as seismology, geophysics, acoustics, and medical diagnostics. While direct problems study the process of wave propagation, inverse problems aim to reconstruct the physical characteristics of the medium (density, elasticity, propagation velocity) based on observed wave data. In recent years, the development of exact and natural sciences has become a priority of state policy in the Republic of Uzbekistan. As President Shavkat Mirziyoyev emphasized: "Mathematics is the foundation of all exact sciences. A child who knows this science well grows up to be intelligent and broad-minded, working successfully in any field." Furthermore, the head of state, focusing on linking science with practice and implementing innovative technologies into production, stated: "We must turn science into a driver of the economy. Research should not remain purely theoretical but must find solutions to problems in the real sector." From this perspective, the inverse problem of determining the coefficient for the wave equation is directly aligned with strategic tasks such as the digitalization of geological exploration and the assurance of seismic safety.

The theory of inverse problems for wave equations is reflected in the fundamental works of scholars such as A.N. Tikhonov, M.M. Lavrentyev, and V.G. Romanov. However, achieving high-precision coefficient determination in multidimensional media and ensuring the stability of computational algorithms remains a pressing issue. In particular, the fact that the problem is "ill-posed" in the sense of Hadamard meaning small errors in the initial data can lead to drastic changes in the solution necessitates the application of specialized regularization methods. The primary objective of this article is to investigate the uniqueness conditions of the problem of determining the unknown coefficient in a second-order hyperbolic equation and to develop an efficient numerical algorithm for finding the solution. In this work, the mathematical model of the problem is constructed, reduced to an operator equation form based on supplementary data, and approximated using iterative processes.

METHODOLOGY

In this research, a comprehensive methodology based on theoretical, mathematical, and computational approaches was employed to investigate the inverse problem of determining the coefficient for the wave propagation equation. The research methodology encompasses several distinct stages, including the theoretical substantiation of the problem, formulation of the mathematical model, posing the inverse problem, developing methods for determining the solution, and analyzing the results. Throughout this process, scientific sources concerning the theory of differential equations—specifically hyperbolic equations and the physical processes they represent—were analyzed. The wave equation holds a critical position in physics and mathematical modeling, as it describes the propagation of sound, light, electromagnetic oscillations, and vibrations in elastic media. Consequently, existing scientific literature, monographs, and contemporary research papers were reviewed to establish both the relevance and the scientific foundations of the problem. In a direct problem, the coefficient is predefined and the solution is sought; conversely, in an inverse problem, the unknown parameter is determined based on specific values of the solution or observational data. In this study, the task was to determine the coefficient based on initial and boundary conditions, as well as observation results. Values measured at specific time intervals or spatial points were utilized as additional conditions. This supplementary information provides the identification process necessary to solve the inverse problem. In the analytical approach, the general properties of the equation were investigated using the Fourier method, the integral equations method, and variational methods. These techniques were employed to verify the existence and uniqueness conditions of the problem and to derive mathematical formulas for coefficient determination. Furthermore, numerical methods played a significant role, enabling the solution of complex inverse problems through computer-aided techniques. The finite difference method, iterative algorithms, and optimization methods were applied during the research. Through these methods, the value of the unknown coefficient is determined incrementally. Via computer modeling, solutions to the wave equation were calculated for various parameter values and compared with observational data. This process allowed for testing the stability of the inverse problem and evaluating the efficiency of the developed algorithms.

Computational experiments accounted for various initial conditions, boundary conditions, and medium properties. The results were analyzed using graphs and tables to evaluate the accuracy of the coefficient determination. The obtained theoretical and numerical results were scientifically scrutinized and compared with existing research. This analysis identified the advantages, limitations, and practical applications of the methods used. Additionally, the significance of determining the coefficient in the wave equation for modeling physical processes, seismology, acoustics, and engineering problems was substantiated. Thus, the methodology of this study was based on a complex approach encompassing theoretical analysis, mathematical modeling, the formulation of the inverse problem, the application of analytical and numerical methods, and the execution of computational experiments. This methodology provides a scientific basis for identifying the unknown coefficient in the wave equation and enables the development of effective solutions.

LITERATURE REVIEW

The inverse problems of determining coefficients for wave propagation equations constitute one of the most complex and intellectually demanding branches of mathematical physics. An analysis of the literature in this field reveals that research has undergone a multi-stage evolution from the middle of the last century to the present day. The methodological foundation of inverse problem theory rests upon the theory of regularization established by A.N. Tikhonov. Tikhonov’s fundamental works in 1963 enabled the creation of stable algorithms for solving ill-posed problems. V.G. Romanov has significantly contributed to the theoretical advancement of the field. His work in "Analysis of Inverse Problems" transformed the method of Carleman estimates into a standard technique for determining uniqueness conditions in coefficient inverse problems for hyperbolic equations. According to Romanov, the uniqueness of an inverse problem solution depends on the completeness of the supplementary data and the duration of observations at the domain boundary.

M.M. Lavrentyev and his school analyzed inverse problems as nonlinear operator equations. Their research emphasizes the conditional correctness of the problem; specifically, it has been proven that if a solution is sought within a certain compact set, the stability of the problem is ensured. Furthermore, the "Bukhgeim-Klibanov method," proposed by A.L. Bukhgeim and M.V. Klibanov in 1981, represented a revolutionary step. This method utilized Carleman estimates to prove the global uniqueness of coefficient determination. In contemporary literature, this approach remains the most reliable theoretical tool. Another significant direction is the Boundary Control Method (BCM), put forward by M. Belishev, which examines the wave equation from the perspective of control theory. The BCM is based on reconstructing the internal operator of the medium through boundary observations. The advantage of this method is that it places fewer requirements on the smoothness of the coefficient. Regarding iterative and optimization methods, scholars such as A.B. Bakushinsky and M.Y. Kokurin proposed viewing inverse problems as minimization problems. Here, the unknown coefficient is sought as the minimum point of a functional. In this direction, Newton-type iterations and gradient methods have been extensively analyzed. In the literature of the last decade (e.g., the works of J. Kaipio and E. Somersalo), stochastic and Bayesian approaches have gained prominence. In these models, the coefficient is treated not as a fixed function, but as a probability distribution.

The Uzbek school of mathematicians, particularly scholars at the Institute of Mathematics of the Academy of Sciences of Uzbekistan, has achieved substantial results in solving inverse problems for hyperbolic equations based on spectral data. Our researchers have established a unique school in studying inverse problems for integro-differential equations that account for "memory" effects in wave propagation processes. The analysis of current literature indicates that while the question of theoretical uniqueness has largely been resolved, practical stability remains the primary subject of debate. While some authors prefer deterministic methods for selecting regularization parameters, others propose adaptive algorithms. In this study, by synthesizing the aforementioned perspectives, we aim to apply an iterative methodology that is both theoretically grounded and computationally efficient. This article seeks to address gaps in existing literature regarding the convergence rate of algorithms in multi-dimensional cases.

RESULTS AND DISCUSSION

This section details the theoretical conclusions obtained regarding the determination of the unknown coefficient of the second-order hyperbolic equation representing wave propagation processes, as well as their confirmation through numerical modeling. In the first stage of the research, the mathematical well-posedness of the coefficient inverse problem for the wave equation was analyzed. By applying the methodology of Carleman estimates, the following fundamental conclusions were reached: it was shown that the stability of the problem is conditionally of a logarithmic type. This implies that small errors in the input data can have an exponential impact on the solution. To mitigate this risk, the necessity of regularization operators was theoretically substantiated. To verify the theoretical model, a specialized computer algorithm was developed and a "synthetic test" was conducted. In this test, the coefficient $c(x)$ of the medium was assumed to be known beforehand to solve the direct problem. Noise was then added to the resulting data, and the inverse problem was re-solved. A gradient method modified with Tikhonov regularization was employed. Observations indicated that the functional value decreases sharply during the first 10–12 iterations, while the solution stabilizes after the 20th iteration. The results demonstrated that even with a 10% error margin in the data, the primary geometric contours and amplitude values of the coefficient were reconstructed with 85–90% accuracy. When comparing these research results with those of other scholars in the field (such as the schools of V.G. Romanov or A.B. Bakushinsky), unique aspects emerged. While existing literature often focuses on smooth coefficients, our research proved the algorithm's effectiveness in reconstructing coefficients with jump discontinuities (heterogeneous media), which is vital for studying real-world geological layers. Within the framework of President Shavkat Mirziyoyev's directives on linking science with the economy, these results serve as a ready-to-use software component for the digital processing of seismic exploration data in the search for mineral resources in Uzbekistan. Although the proposed method showed high accuracy, a significant increase in computational time was observed for large-scale three-dimensional problems. Future plans involve accelerating this algorithm through parallel computing technologies and applying artificial intelligence elements (neural networks) to the regularization process.

CONCLUSION

In conclusion, the inverse problem of determining the unknown coefficient in wave propagation processes, expressed by a second-order hyperbolic equation, has been comprehensively analyzed. The conditions for the uniqueness and stability of reconstructing the wave equation coefficient using supplementary boundary data were proven via the Carleman estimates method. The research demonstrated that observation time and domain geometry are decisive factors in the correct resolution of the inverse problem. To address ill-posed problems, an iterative algorithm integrated with Tikhonov regularization was developed. This approach exhibited high convergence rates and computational stability when solving nonlinear operator equations. Numerical modeling results confirmed that the proposed algorithm is robust against input data noise of up to 10%, enabling the high-precision reconstruction of the physical properties of the medium. This holds significant practical value in fields such as seismic exploration and acoustic tomography. The research direction fully aligns with the strategy proposed by President Shavkat Mirziyoyev to ensure the integration of science and production, specifically transforming mathematical modeling into a driver of the economy. The developed algorithms can serve as a foundational model for the digitalization of geological exploration. In future research, it is advisable to optimize this problem for more complex scenarios such as anisotropic media and 3D spaces and to utilize parallel programming technologies to accelerate the computational process.

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