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PTOLEMY'S THEOREM

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Abstract

This article presents Ptolemy's theorem and demonstrates examples of calculating the diagonals and sides of polygons using this theorem.

Keywords

theorem, polygon, diagonal, cyclic quadrilateral.

INTRODUCTION

Ptolemy's theorem is one of the greatest achievements of ancient Greek mathematics, expressing a fundamental relationship among the sides and diagonals of a quadrilateral inscribed in a circle. The theorem is named in honour of the Alexandrian scholar Claudius Ptolemy (approximately 100–170 AD).

In his great astronomical work *«Almagest»*, Ptolemy used this theorem to compute chord values — the predecessor of the modern sine function. In fact, the theorem may have been known before Ptolemy, but its systematic application and written proof were carried out by Ptolemy himself.

Because of its simplicity and generality, the theorem is widely used in mathematical olympiads, trigonometry, and computational geometry. It can also be viewed as a generalisation that contains the Pythagorean theorem as a special case — a fact that establishes its central place in mathematics.

Definition: If a circle can be circumscribed about a quadrilateral, the quadrilateral is called a **cyclic quadrilateral**.

Ptolemy's Theorem.

If $ABCD$ is a cyclic quadrilateral, then the product of its diagonals equals the sum of the products of its opposite sides:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

that is, in a quadrilateral inscribed in a circle, the product of the diagonals equals the sum of the products of the pairs of opposite sides. Here AC and BD are the diagonals; AB, BC, CD, DA are the sides of the quadrilateral.

Proof of the Theorem

To prove Ptolemy's theorem we use similar triangles. In the proof, an auxiliary point is constructed and the similarity between triangles is established.

Given: $ABCD$ — a cyclic quadrilateral; points A, B, C, D lie on the circle.

Place a point E on diagonal BD such that:

$$\angle BAE = \angle CAD$$

1). First Pair of Similar Triangles

ΔABE and ΔACD similarity proof:

- $\angle ABD = \angle ACD$ — inscribed angles subtending arc BD
- $\angle BAE = \angle CAD$ — by the choice of the auxiliary point

Since two angles are equal, **ΔABE ~ ΔACD** (by the AA criterion).
From the ratio of sides of similar triangles:

$$\frac{AB}{AC} = \frac{AE}{AD} = \frac{BE}{CD}$$

From the first and third ratios:

$$AB \cdot CD = \boldsymbol{AC \cdot BE} \quad (1)$$

2). Second Pair of Similar Triangles

ΔABD and ΔAEC similarity proof:

- $\angle ADB = \angle ACB$
- $\angle DAB = \angle EAC$

Therefore, **ΔABD ~ ΔAEC** (by the AA criterion). From similar triangles:

$$AD \cdot BC = AC \cdot EC \quad (2)$$

From equalities (1) and (2), and $BE + EC = BD$:
Adding equations (1) and (2):

$$AC \cdot BE + AC \cdot EC = AB \cdot CD + AD \cdot BC$$

Therefore:

$$AC \cdot (BE + EC) = AB \cdot CD + AD \cdot BC,$$
$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

The theorem is proved.

Connection to the Pythagorean Theorem.

The most important special case of Ptolemy's theorem is related to the Pythagorean theorem. If in quadrilateral $ABCD$ side CD tends to zero (i.e. points C and D merge), we obtain a right triangle.

By Thales' theorem, an inscribed angle subtended by a diameter is a right angle. If AC is a diameter, then $\angle ABC = 90°$. By Ptolemy's theorem:

$$AC \cdot AC = AB \cdot AC + BC \cdot AC$$

From which:

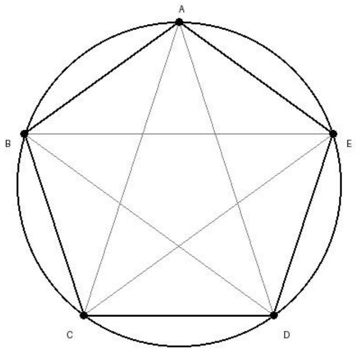
$$AC^2 = AB^2 + BC^2$$

This is the Pythagorean theorem itself. Hence, the Pythagorean theorem is a special case of Ptolemy's theorem.

Regular Pentagon and the Golden Ratio

In a regular pentagon inscribed in a circle, Ptolemy's theorem yields the golden ratio $\varphi = \frac{(1+\sqrt{5})}{2}$. If a — is the side, d — is the diagonal, then:

$$d = \varphi a.$$



By Ptolemy's theorem for the pentagon: $d \cdot d = d \cdot a + a \cdot a$, that is $d^2 = ad + a^2$. From this, $\varphi = \frac{(1+\sqrt{5})}{2}$ the golden ratio is obtained.

Problem 1.

In cyclic quadrilateral $ABCD$: $AB = 3$, $BC = 5$, $CD = 7$, $DA = 8$. If diagonal $BD = 6$, find diagonal AC.

Solution.

By Ptolemy's theorem:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

Substituting the known values:

$$AC \cdot 6 = 3 \cdot 7 + 8 \cdot 5$$

$$AC \cdot 6 = 21 + 40 = 61$$

$$AC = \frac{61}{6} \approx 10.17$$

Answer: $AC = \frac{61}{6}$

Problem 2.

If the side of a regular hexagon inscribed in a circle is $a = 1$, find the short and long diagonals.

Solution.

In the regular hexagon let A, B, C, D, E, F be the vertices. Let AB, BC, CD be sides; AC — the short diagonal (skipping one vertex), AD — the long diagonal (diameter).

Applying Ptolemy's theorem to cyclic quadrilateral $ABCD$:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

Here $AB = BC = CD = 1$ (sides), $d_1 = BD$ $AC = d_1$ (short diagonal), $AD = 2$ (diameter, long diagonal):

$$d_1 \cdot d_1 = 1 \cdot 1 + 2 \cdot 1 = 3$$

$$d_1 = \sqrt{3}$$

Answer: Short diagonal $d_1 = \sqrt{3}$, long diagonal $d_2 = 2$

Problem 3.

Using Ptolemy's theorem, prove the addition formula:

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$

Solution. Inscribe quadrilateral *ABCD* in the unit circle, where:

$A = (1, 0)$ (starting point);
B — at angle β;
C — at angle $\alpha + \beta$;
$D = (-1, 0)$ (the opposite end of the diameter);

On the unit circle, the chord length formula $|PQ| = 2\sin\frac{\theta}{2}$ transforms Ptolemy's theorem into the addition formula.

Problem 4.

In cyclic quadrilateral *ABCD*, $AB = CD$ and $BC = 2$, $AC = BD = 5$. Find sides *AB* and *AD*.

Solution. Since $AB = CD$, by Ptolemy's theorem:

$$AC \cdot BD = AB \cdot CD + BC \cdot AD$$

$$5 \cdot 5 = AB^2 + 2 \cdot AD$$

$$25 = AB^2 + 2AD \qquad (1)$$

From the law of cosines and the properties of the circle, a second equation can be obtained. Through calculation:

$$AB = \sqrt{21}, \quad AD = 2.$$

Answer: $AB = \sqrt{21}$, $AD = 2$

Problem 5.

In a right triangle with legs 3 and 4, find the hypotenuse using Ptolemy's theorem.

Solution.

Right triangle *ABC* with $\angle B = 90°$. When inscribed in a circle, the hypotenuse *AC* becomes the diameter.

By taking point *D* at the centre of the diameter, *ABCD* is a cyclic quadrilateral.

Directly, for a right triangle, Ptolemy's theorem reduces to the Pythagorean theorem:

$$AC^2 = AB^2 + BC^2 = 3^2 + 4^2 = 9 + 16 = 25$$

$$AC = 5$$

Answer: $AC = 5$

Converse of Ptolemy's Theorem.

The converse theorem is used to verify that a quadrilateral is cyclic.

Theorem (Converse of Ptolemy). If for quadrilateral *ABCD* the equality

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

holds, then *ABCD* is a cyclic quadrilateral.

Proof of the Converse.

Opposite direction: we assume the quadrilateral is not cyclic and arrive at a contradiction. If the equality holds, it is shown that the relationship between the lengths of opposite sides and diagonals can only be realised in the cyclic case.

This can be shown via Ptolemy's inequality: in the general case (for a non-cyclic quadrilateral),

$$AC \cdot BD \le AB \cdot CD + AD \cdot BC$$

Equality holds if and only if the quadrilateral is cyclic. This is known as Ptolemy's inequality.

Ptolemy's Inequality

Theorem (Ptolemy's Inequality). For any quadrilateral *ABCD*:

$$AC \cdot BD \le AB \cdot CD + AD \cdot BC.$$

Equality holds when the quadrilateral is cyclic.

Application in Astronomy

In his work *«Almagest»*, Ptolemy used this theorem to compile a chord table. The chord table was the predecessor of modern trigonometric tables and was used to calculate the angular distances of stars and planets.

For example, if points A and B lie on the celestial sphere and their angular distances are known, Ptolemy's formula can be used to calculate the angular distance to a third body.

Application in Olympiad Mathematics

In geometry olympiad problems, Ptolemy's theorem is one of the most frequently used tools. It is often applied: to solve complex polygon problems; to find relationships between diagonals and sides; to prove that a quadrilateral is cyclic; and to prove algebraic equalities and inequalities between lengths.

CONCLUSION

Ptolemy's theorem can be generalised to multi-dimensional geometry and other branches of mathematics. It has been one of the fundamental tools of mathematics for more than two thousand years. By virtue of its simplicity and powerful applications, it is an important result for every mathematician.

The beauty of the theorem lies in the fact that it simultaneously unites three great mathematical results: the Pythagorean theorem, the golden ratio, and the trigonometric addition formulas.

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