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"FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS AND THE METHOD FOR SOLVING THE CAUCHY PROBLEM ASSOCIATED WITH THEM"

Doniyor G'afurovich Rayimov
PhD, Associate Professor,
Department of General Technical Sciences,
Asia International University

Annotation: In this topic, we study first-order partial differential equations, including their quasilinear and linear forms, as well as homogeneous and non-homogeneous types. We also discuss methods for constructing both particular and general solutions of these equations using their characteristic equations and families of characteristic curves, known as first integrals. Furthermore, we examine how to determine the solution of the Cauchy problem posed for such equations based on the obtained general solution.

Introduction The problem of finding the general solution of a first-order quasilinear partial differential equation is reduced to solving a homogeneous differential equation, and the general solution can be obtained by the method of characteristics. The solution of the Cauchy problem is then found by checking the additional condition of the Cauchy problem along the characteristics and introducing a new variable if necessary

Definition 1. $u = u(x_1, x_2, \cdots, x_n)$ differential equation involving an unknown function, its independent variables, and its first-order partial derivatives is called a first-order partial differential equation, and in general form it is written as

$$F(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}) = 0 \qquad (1)$$

It is written in the following form. Here, Fis a given function.

For example, the following

$$xu_x + u_y = 0, \quad u\sin(u_x - 2xyu_y) = xu_x, \quad u_x + u_y u_z + zu = x + yz, \qquad x^2 u_x + zy^2 u_y + 5u_z = xz$$

The following equations are examples of first-order partial differential equations.

Definition 2. If (1) is a first-order partial differential equation

$$b_1(x_1, \cdots, x_n, u)\, u_{x_1} + \cdots + b_n(x_1, \cdots, x_n, u)\, u_{x_n} = f(x_1, \cdots, x_n, u) \qquad (2)$$

is of the following form, then it is called a first-order quasilinear differential equation. Here, fis a given function.

If in equation (2) $f(x_1, \cdots, x_n, u) = 0$ then it is called homogeneous; otherwise, that is, if in the domain under consideration $f(x_1, \cdots, x_n, u) \quad 0$ then equation (2) is called a non-homogeneous first-order quasilinear partial differential equation.

If in equation (2) the participating $b_i, i=1,2,\cdots,n$ coefficients depend only on $(x_1,x_2,\cdots,x_n)$ certain variables and do not depend on the unknown function u , and $f(x_1,\cdots,x_n,u)$ if the equation depends linearly on the unknown function uas well, then such an equation is called a first-order linear partial differential equation.

For example, among the examples given above, the first and the fourth are first-order linear partial differential equations. The first of these is a homogeneous equation with respect to a function of two variables, whereas the fourth is a non-homogeneous equation with respect to a function of three variables.

The method for finding the general solution of a first-order homogeneous partial differential equation and the solution of the corresponding Cauchy problem.

Let us assume that we are given a first-order homogeneous linear partial differential equation:

$$b_1(x_1,\cdots,x_n)u_{x_1}+\cdots+b_n(x_1,\cdots,x_n)u_{x_n}=0. \qquad (3)$$

In this case $b_i(x_1,\cdots,x_n),\ i=1,2,\cdots,n$ the coefficients are functions of certain $D \quad R^n$ functions defined in a certain domain, which, together with their first-order partial derivatives, are continuous and nowhere simultaneously equal to zero. For clarity $b_n(x_1,x_2,\cdots,x_n) \quad 0$ Let them be as such.

Usually, along with (3), the following system of differential equations, called its characteristic equations, is considered:

$$\frac{dx_1}{b_1(x_1,\cdots,x_n)}=\frac{dx_2}{b_2(x_1,\cdots,x_n)}=\cdots=\frac{dx_n}{b_n(x_1,\cdots,x_n)}. \qquad (4)$$

System (4) is replaced by the following system of equations, which is equivalent to it

$$\frac{dx_i}{dx_n}=\frac{b_i(x_1,\cdots,x_n)}{b_n(x_1,\cdots,x_n)},\ i=1,2,\cdots,n-1. \qquad (5)$$

$b_i(x_1,\cdots,x_n),\ i=1,2,\cdots,n$ under the conditions imposed on the coefficients above, system (5) has n−1linearly independent first integrals:

$$\varphi_i(x_1,x_2,\cdots,x_n)=C_i,\ \ i=1,2,\cdots,n-1. \qquad (6)$$

equation (6) is called the family of characteristic curves of equation (3). The following lemma establishes the relationship between the solutions of equation (3) and those of system (5).

Lemma 1) If $\varphi(x_1,x_2,\cdots,x_n)=C$ is a first integral of (5), then $u=\varphi(x_1,x_2,\cdots,x_n)$ the function is a particular solution of differential equation (3).

2) If $u=\varphi(x_1,x_2,\cdots,x_n)$ is a particular solution of (3), then $\varphi(x_1,x_2,\cdots,x_n)=C$ it forms a family of first integrals of (5).

3) If $\varphi_i(x_1, x_2, \cdots, x_n) = C_i,\ i = 1,2,\cdots,n-1$ are first integrals of (5), then they constitute the general solution of differential equation (3)

$$u = F(\varphi_1, \varphi_2, \cdots, \varphi_{n-1})$$

consisting of them. Here, in the domain under consideration, F an arbitrary function that is continuously differentiable.

Example 1. $xu_x + yu_y + zu_z = 0$ Find the general solution of a first-order linear partial differential equation. Write several particular solutions.

Solution. The system of characteristic equations corresponding to this equation takes the following form:

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}.$$

This system is

$$\frac{dx}{x} = \frac{dz}{z}$$
$$\frac{dy}{y} = \frac{dz}{z}$$

written in the following form, and its integrals are then determined:

$$\begin{matrix} \ln x = \ln z + \ln C_1 \\ \ln y = \ln z + \ln C_2 \end{matrix} \quad \text{yoki} \quad \begin{matrix} \frac{x}{z} = C_1 \\ \frac{y}{z} = C_2 \end{matrix}.$$

Then, according to the theorem, the general solution of the given equation is expressed through an arbitrary continuously differentiable function Fas follows

$$u = F\ \frac{x}{z}, \frac{y}{z}$$

in the following form. From this form of the general solution, we can write several particular solutions of the given equation:

$$1)\ \ u = \frac{x}{z} + \frac{y}{z} \qquad 2)\ u = \frac{x}{z} + \sin\frac{y}{z} \qquad 3)\ u(x,y,z) = \ \frac{x}{z}^{\ 2} + \cos^2\frac{y}{z} + 1.$$

Calculation results

As can be seen from these examples, a first-order linear partial differential equation has infinitely many solutions. The important question of both physical and mathematical significance is: what additional condition should be imposed so that this equation has a unique solution.

Thus, we have become acquainted with first-order partial differential equations, their various types, the method of characteristics for finding their general solution, and the way to solve the corresponding Cauchy problem. That is, a first-order linear partial differential equation has a unique solution when a single additional condition is imposed.

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