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GENERAL SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS AND

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Abstract. This paper analyzes the general solutions of linear equation systems and the conditions for their existence. The study employs the Kronecker–Capelli theorem, matrix rank, and the Gauss elimination method as primary theoretical tools. The affine structure of the solution space and the relationship between the number of free variables and the dimension of the solution space are examined in detail. The results allow a comprehensive understanding of the solution structure from both algebraic and geometric perspectives.

Keywords: Linear equations, general solution, Kronecker–Capelli theorem, matrix rank, Gauss elimination, affine space, free variables, linear independence, augmented matrix, particular solution, homogeneous solution, solution space structure.

Kirish. Linear systems of equations constitute the basic constructive element of modern mathematical modeling and serve as a universal tool for providing a quantitative description of multidimensional processes. In fields such as mechanics, structural engineering, power engineering, automation, economics, and information technology, the state of complex objects is frequently expressed through linear relationships among several variables. In the analysis of such systems, not only the search for an individual solution but also the determination of the structure of the set of all possible solutions is regarded as an important scientific problem. Consequently, establishing the general solution of a linear system of equations holds significant theoretical and practical importance.

The existence or non-existence of solutions in linear systems is directly linked to their algebraic structure, in which the properties of the coefficient matrix play a decisive role. In particular, the concepts of matrix rank, linear independence, basis, and dimension serve as a fundamental theoretical foundation for determining the conditions under which solutions exist. Through these concepts, it is possible to rigorously establish, on the basis of precise mathematical criteria, whether a system of equations is consistent or inconsistent and whether its solutions are unique or infinitely numerous. This further enhances the logical coherence of linear algebra theory and underscores its practical significance.

In contemporary scientific literature, the general solution of a linear system of equations is commonly interpreted from the perspective of linear space theory. In this approach, the set of solutions is viewed geometrically as an affine space, with its dimension determined by the number of free variables. In particular, the relationship between homogeneous and non-homogeneous systems, along with the representation of the general solution as the sum of a particular solution and the space of homogeneous solutions, yields important theoretical results. These viewpoints enable a deeper understanding of the internal structure of linear systems.

In this article, the general solutions of linear systems of equations and the conditions for their existence are investigated using a comprehensive approach. During the study, the Kronecker–Capelli theorem, the rank of matrices, and the Gauss elimination method are employed as the primary theoretical tools. The main objective of the article is to analyze the structure of solutions of linear systems on a rigorous mathematical basis, to present their algebraic and geometric interpretations in a unified manner, and to reveal the importance of these theoretical results in practical problems.

Methods and Materials. This study is focused on the general analysis of linear systems of equations, with primary attention directed toward determining the existence, number, and structural characteristics of solutions. Within the scope of the research, systems in which the number of unknowns is less than, equal to, or greater than the number of equations were classified into separate categories, and the algebraic properties of the solutions were generalized for each case. All analyses were carried out in the field of real numbers; however, to ensure the universality of the results, the possibility of extending them to the field of complex numbers was also theoretically substantiated.

The central element of the research methodology is linear space theory, in which the concepts of vector space, subspaces, basis, and dimension were employed as the main theoretical tools. The relationship between the column space and row space of the coefficient matrix, the equality of their dimensions, and the influence of these dimensions on the existence of solutions for the system were analyzed in depth. In particular, the invariance of the rank concept and its property of being preserved under elementary transformations were widely used as a methodological foundation.

The augmented matrix concept served as the primary analytical tool for determining the consistency or inconsistency of a linear system of equations. Using this approach, the possibility of expressing the vector of free terms as a linear combination of the coefficient columns was examined. As a result, the Kronecker–Capelli theorem was interpreted not merely as a formality, but as a theoretical mechanism that reveals the internal algebraic structure of the system.

In the computational process, along with the Gauss elimination method, the Gauss–Jordan algorithm (reduction to reduced row echelon form) was also utilized. This approach simplifies the identification of basic and free variables and enables the general solution to be expressed in parametric form. At the same time, issues of algorithmic complexity and computational stability were methodically considered, and the limits of applicability of these methods for large-scale systems were discussed.

A separate methodological direction was devoted to illuminating the structural relationship between homogeneous and non-homogeneous linear systems. It was substantiated from both geometric and algebraic perspectives that the solution space of a homogeneous system forms a linear subspace, while the solutions of a non-homogeneous system are represented as an affine space parallel to this subspace. This approach made it possible to explain the spatial interpretation of the general solution more precisely.

As an additional method, the dimension theorem and the relationship between the dimension of the solution space and the number of free variables were used to determine the dimension of the solution space. This allowed the degree of freedom of the system to be expressed on a rigorous mathematical basis. In the process of generalizing the research results, conclusions obtained for different types of systems were compared, and the possibilities of their application in practical problems were analyzed.

Results. The theoretical analyses conducted have established that the existence, number, and structural characteristics of solutions to a linear system of equations are strictly dependent on the algebraic parameters of the coefficient matrix. In particular, the rank of the matrix is one of the main invariants of the system; it remains unchanged under elementary row and column operations and manifests itself as a fundamental criterion that determines whether the system has solutions or not. The research results once again confirmed the possibility of applying the Kronecker–Capelli theorem in its general form.

The results show that if the ranks of the coefficient matrix and the augmented matrix are equal, the system is consistent, and its set of solutions forms a non-empty affine space. In this case, the dimension of the solution space is determined by the difference between the number of unknowns n n n and the rank of the matrix r r r, i.e., n−r n - r n−r. This parameter is interpreted as the number of free variables and mathematically describes the structural complexity of the solution space.

The interrelationship between homogeneous and non-homogeneous systems was analyzed in greater depth. The study revealed that the general solution can be expressed in the following form:

$$x=x_0+\sum_{i-1}^{k}\alpha_i v_i$$

where x_0 is a particular solution, v_i are linearly independent solutions of the homogeneous system, α_i *are* parameters, and $k=n-r$. This formula theoretically substantiates the structure of the set of solutions as an affine space and ensures algebraic-geometric consistency.

The results of the geometric analysis indicate that the mutual arrangement in space of the hyperplanes represented by the linear system is the main factor determining the existence and number of solutions. In consistent systems, the intersection space of these hyperplanes exists, and its dimension fully corresponds to the algebraic results. This strengthens the theoretical connection between linear algebra and spatial interpretation, and also allows for a deeper explanation of the affine nature of the solution space.

The results of computational experiments showed complete agreement with the outcomes obtained using the Gauss and Gauss–Jordan elimination methods. The structures of the basic and free variables identified during the reduction to row echelon form fully align with the theoretical conclusions and precisely express the parametric structure of the solution space.

Discussion. The obtained results indicate that the solution space of a linear system of equations is directly related not only to its algebraic parameters but also to the internal structural characteristics of the system. The difference between the rank of the coefficient matrix and the rank of the augmented matrix strictly determines, from the standpoint of the Kronecker–Capelli theorem, whether the system has solutions or not. At the same time, the strict relationship between the number of free variables and the dimension of the solution space determines the optimization of computations in high-dimensional systems, thereby enabling the integration of algebraic and computer-based computational approaches.

Geometric interpretations make it possible to understand the results more deeply. In consistent systems, the intersection of hyperplanes in space forms an affine space, and its dimension fully coincides with the algebraic calculations. This shows that the difference between the solution spaces of homogeneous and non-homogeneous systems is determined solely by the combination of a particular solution and the solutions expressed through homogeneity. As a result, the structuring of the general solution as an affine space is accepted as a single concept from both algebraic and spatial perspectives.

In addition, during the study, the issues of stability and sensitivity of the solution space were also analyzed. From an algorithmic standpoint, in systems with a high number of free variables, the problems of numerical errors and computational stability become relevant, and it was determined that specially optimized Gauss–Jordan algorithms and matrix transformations must be applied. This result is of great importance not only theoretically but also in practical engineering calculations, as it reduces computational costs and increases accuracy when solving large-scale and sparse matrix systems.

The results also show that identifying the affine nature of the solution space of a linear system of equations requires not only the integration of algebraic and geometric approaches but also creates the necessary methodological foundation for obtaining high-precision results in real engineering and scientific modeling problems. Furthermore, the results provide the opportunity to apply the obtained concepts to large-scale, complex, and multi-parameter systems, thereby forming an integrated approach that combines linear algebra, numerical methods, and engineering computations.

Conclusion. The results of the conducted research show that the solutions of a linear system of equations are not limited solely to algebraic parameters and the properties of the coefficient matrix, but are also directly related to the internal structural and spatial characteristics of the system. Through the Kronecker–Capelli theorem, the existence or inconsistency of solutions for the system is rigorously determined, and the mathematical relationship between the dimension of the solution space and the number of free variables is confirmed. At the same time, the affine structure of the solution space constitutes a single concept that is consistent from both algebraic and geometric perspectives.

During the study, the difference between homogeneous and non-homogeneous systems was clearly identified through the representation of the general solution as a combination of a particular solution and homogeneous solutions. This result ensures the analysis of the set of solutions as an affine space not only from a theoretical standpoint but also in practical engineering problems, particularly in mechanics, electrodynamics, optimization, and economic modeling.

The computational analyses show that in systems with a high number of free variables, issues of numerical stability are relevant, and the application of optimized algorithmic approaches is necessary. This makes it possible to increase accuracy and reduce computational costs when solving large-scale and sparse matrix systems. At the same time, the results once again confirm the organic connection between the theory of linear algebra and practical computations.

In conclusion, the research has made it possible to analyze the general solutions of a linear system of equations on a rigorous mathematical basis and to determine their spatial structure. The obtained results are not only of theoretical importance but also hold practical significance for high-precision numerical modeling, the optimization of engineering systems, and the solution of complex scientific problems. In addition, it has been scientifically proven that the research methodology and approaches can be extended to large-scale and complex multi-parameter systems.

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