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QUALITATIVE PROPERTIES OF SOLUTIONS OF DOUBLE NONLINEAR PARABOLIC SYSTEM WITH SOURCE

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Abstract. This paper investigates a doubly nonlinear parabolic system of equations used to model heat conduction, filtration processes, nonlinear diffusion, and biological population dynamics. The study focuses on the existence and qualitative properties of weak solutions, as well as localization phenomena and the formation of nonstationary dissipative structures. The obtained results contribute to the understanding of nonlinear processes in heterogeneous media and the development of methods for analyzing degenerate parabolic systems.

Keywords. doubly nonlinear parabolic system, weak solution, nonlinear diffusion, localization, dissipative structures.

This paper Consider the qualitative properties of solutions to the Cauchy problem for the following system of nonlinear reaction-diffusion equations in the domain $Q=\left\{(t,x):t>0,x \quad R^N\right\}$

$$L_1(u,v) \quad -\frac{\partial u}{\partial t}=div\left(u^{m_1-1}\left| \quad u^k\right|^{p-2} \quad u\right)+\varepsilon u^{p_1}v^{q_1}=0, \quad \varepsilon=\pm 1$$

$$L_2(u,v) \quad -\frac{\partial v}{\partial t}=div\left(v^{m_2-1}\left| \quad v^k\right|^{p-2} \quad v\right)+\varepsilon u^{p_2}v^{q_2}=0 \qquad (1)$$

$$u(0,x)=u_0(x) \quad 0, \qquad v(0,x)=v_0(x) \quad 0, \quad x \quad R^N \qquad (2)$$

where, $k \quad R, m_1,m_2>1, p_i,q_i \quad 1, p \quad 2$ positive real numbers, $\quad (\ )=grad(\ )$ and $u_0(x) \quad 0, v_0(x) \quad 0$, is a non-trivial, non-negative, bounded and sufficiently smooth function.

The paper is devoted to studies of solutions of double nonlinear parabolic system equation describing processes of heat conductivity, filtration in liquid and gas, nonlinear diffusion, biological population in two componential mediums with source or absorption [4]. It is interesting to study unusual properties of the system -localization and the associated emergence of nonstationary dissipative structures.

Problem (1) - (2) appears in various applications [1-4] (see [4] and literature therein). In particular, this problem describes different physical, diffusion, chemical, biological and other processes in two componential nonlinear mediums.

A feature of this system of equations is the presence of degeneracy. In the region where the system of equations (1) degenerates into a first-order equation. Therefore, it is necessary to study a weak solution, since in this case solutions (1)-(2) may not exist in the classical sense.

This work in practical numerical parameters for one equation and system has been studied by many authors[7-12].

For example at first Fujita considered the following initial value problem [7]:

$$u_t=\Delta u+u^p \qquad (x,t) \quad ^N \quad (0,T)$$

$$u(x,0)=u_0(x) \quad 0 \quad x \quad ^N \qquad (3)$$

(Δ denotes the N-dimensional Laplace operator)

$u_0(x)$ is a bounded nonnegative continuous function, $u_0(x) \quad 0$ where p > 1 and T (the length of the existence interval).

He proved the following result concerning nonnegative solutions:

If $1 < p < 1 + \frac{2}{N}$ then all nonnegative solutions blow up in finite time with any nontrivial initial values.

If $p > 1 + \frac{2}{N}$ then global, nontrivial nonnegative solutions exist with sufficiently small initial values.

The case $p = 1 + \frac{2}{N}$ belongs to case (a) but this was proved later. The number $p = 1 + \frac{2}{N}$ is called the critical blow up exponent. Case (a) is called the global nonexistence case (the blow up case), while case (b) is called the global existence case ($T =$). Whenever case (a) happens in a single equation of the initial value problem, there exists a finite number T ($0 < T <$) such that

$$\limsup_{t \to T^-} \sup_{x \quad ^N} u(x,t) \ = \quad .$$

M. Escobedo and M. A. Herrero [5] studied the initial value problem for the system

$$\begin{aligned} u_t &= \ u + u^{p_2} \\ v_t &= \ v + v^{p_2} \end{aligned} \qquad (x,t) \quad ^N \quad (0,T) \qquad (4)$$

$$u(x,0) = u_0(x) \quad 0 \quad v(x,0) = v_0(x) \quad 0, \quad x \quad ^N \quad p_1 > 0, \quad p_2 > 0$$

Their interest was in nonnegative solutions which, for fixed t, decay at infinity.

H. A. Levine [6] studied nonnegative solutions of the initial boundary value problem for the system:

$$u_t = \ u + u^{p_2}, \quad v_t = \ v + v^{p_2} \quad u = v = 0 \qquad (x,t) \quad D \quad (0,T) \qquad (5)$$

$$u(x,0) = u_0(x) \quad 0, \quad v(x,0) = v_0(x) \quad 0 \quad x \quad D, \quad p_1, p_2 \quad 1 \text{ with } p_1, p_2 > 1$$

where D is a cone or the exterior of a bounded domain.

A cone with vertex at the origin is a set of the form

$$\left\{ x \quad ^N \,||x| = r,\ x = \left(r, \underset{\sim}{\theta}\right),\ \underset{\sim}{\theta} \quad \Omega \quad S^{N-1} \right\}$$

where Ω is a region on the unit sphere $S^{N-1} = \left\{ x \quad ^N \,||x| = 1 \right\}$. Then D is a cone with vertex at the origin if and only if we can write $D = (0, \quad) \quad \Omega$.

This problem has also been studied by many authors in the special case when $p = 2$. In particular, Samarskiy A.A., Kurdyumov S.P. and others [1-4] obtained the condition for the global solvability of the Cauchy problem for a degenerate parabolic system.

$$\frac{\partial u}{\partial t} = div\left(u^{m_1-1} \quad u\right) + v^{q_1},$$
$$\frac{\partial v}{\partial t} = div\left(v^{m_2-1} \quad v\right) + u^{p_2} \qquad (6)$$

Self-similar analysis of the solution of the problem (1) studied in [6].

The existence of a unique viscous solution was proved, and in [7] the existence and uniqueness of the classical solution to the Cauchy problem

$$u_t = u^{\alpha_1}\left(u_{xx} + a\,v\right), \quad x \quad \Omega, t > 0,$$
$$v_t = v^{\alpha_2}\left(v_{xx} + b\,u\right), \quad x \quad \Omega, t > 0,$$
$$\mathrm{u} \quad \mathrm{v}|_{\partial\Omega} \quad 0, \quad t > 0,$$
$$\mathrm{u}\left(x,0\right) \quad u_0\left(x\right), \quad v\left(x,0\right) \quad v_0\left(x\right), x \quad \Omega.$$

The local existence of positive classical solutions is proved: for {a, b} ≤ λ_1 there are positive classical solutions, for min {a, b} > λ_1 there are no positive classical solutions.

In this work, using self-similar analysis of solutions we study Fujita type global solvability of solutions problem, estimate of solutions, an asymptotic self-similar solutions established. Based it the numerical analysis of solutions discussed. It is suggested an appropriate initial approximation keeping nonlinear properties of solutions. The influence of the parameters of the reaction-diffusion system on the evolution process investigated.

Fujita type global solvability:

Consider the functions

$$\mathrm{u}_+\left(\mathrm{t},\mathrm{x}\right) = \bar{u}(\mathrm{t})\bar{\mathrm{f}}\left(\xi\right), \quad \mathrm{v}_+\left(\mathrm{t},\mathrm{x}\right) = \bar{v}(\mathrm{t})\bar{\varphi}\left(\xi\right)$$

$$\bar{f}(\xi) = (a - \xi^{\gamma})^{n_1}, \bar{\varphi}(\xi) = (a - \xi^{\gamma})^{n_2}, \gamma = \frac{p}{p-1}, \ \xi = \frac{|x|}{\left(\tau\left(t\right)\right)^{1/p}}; \qquad (7)$$

$$\bar{u}(t) = \quad T + {}_{0}^{t}\,\gamma_1(y)dy \quad{}^{-\alpha 1}, \qquad \bar{v}(t) = \quad T + {}_{0}^{t}\,\gamma_1(y)dy \quad{}^{-\alpha 2}$$

$$\tau\left(\mathrm{t}\right) = {}_{0}^{\mathrm{t}}\,\bar{\mathrm{u}}^{\mathrm{m}_1+\mathrm{k}(\mathrm{p}-2)-1}(\eta)\,\mathrm{d}\eta = {}_{0}^{\mathrm{t}}\,\bar{\mathrm{v}}^{\mathrm{m}_2+\mathrm{k}(\mathrm{p}-2)-1}(\eta)\,\mathrm{d}\eta, \qquad (8)$$

$$\text{if } \mathrm{m}_\mathrm{i} + \mathrm{k}(\mathrm{p}-2) - 1 > 0, \ \mathrm{i} = 1, 2.$$

where

$$\alpha_1 = \frac{(q_2+1) - q_1}{(p_1-1)(q_2-1) - p_2 q_1}, \quad \alpha_2 = \frac{\left(p_1+1\right) - p_2}{(p_1-1)(q_2-1) - p_2 q_1}, \qquad (9)$$

$$n_1 = \frac{(p-1)(d_2+q_1)}{d_1 d_2 - q_1 p_2}, \quad n_2 = \frac{(p-1)(d_1+q_2)}{d_1 d_2 - q_1 p_2}, \qquad (10)$$

Numbers n_1, n_2 are solutions of the system algebraic equation

$$\begin{aligned} d_1 n_1 - q_1 n_2 &= p-1 \\ -p_2 n_1 + d_2 n_2 &= p-1 \end{aligned} \qquad (11)$$

where

$$d_1 = m_1 + k\left(p-2\right) - p_1, \;\; d_2 = m_2 + k\left(p-2\right) - q_2$$

Notice that system (1) have the following self-similar system equation.

$$u(t,x) = \left(T + \int_0^t \gamma_1(y)dy\right)^{-\alpha_1} w(\tau(t),x), \;\; v(t,x) = \left(T + \int_0^t \gamma_1(y)dy\right)^{-\alpha_2} z(\tau(t),x), \qquad (12)$$

$$w(\tau(\mathrm{t}),\mathrm{x}) = \mathrm{f}(\xi), \;\; \mathrm{z}(\tau(\mathrm{t}),\mathrm{x}) = \varphi(\xi)$$

where α_1, α_2 are the numbers determined above and the functions $\mathrm{f}(\xi), \varphi(\xi)$ satisfy

$$L_3(f,\varphi) \quad \xi^{1-N} \frac{d}{d\xi} \; \xi^{N-1} f^{m_1-1} \left|\frac{df^k}{d\xi}\right|^{p-2} \frac{df}{d\xi} + \frac{\xi}{p}\frac{df}{d\xi} + \frac{\alpha_1}{1-\mu_1\alpha_1} \frac{(q_2+1)-q_1)}{(p_1-1)(q_2-1)-p_2 q_1)} f + \varepsilon f^{p_1}\varphi^{q_1} = 0,$$

$$L_4(f,\varphi) \quad \xi^{1-N} \frac{d}{d\xi} \; \xi^{N-1} \varphi^{m_2-1} \left|\frac{d\varphi^k}{d\xi}\right|^{p-2} \frac{d\varphi}{d\xi} + \frac{\xi}{p}\frac{d\varphi}{d\xi} + \frac{\alpha_2}{1-\mu_2\alpha_2} \frac{(p_2+1)-p_1}{(p_1-1)(q_2-1)-p_2 q_1)} \varphi + \varepsilon f^{p_2}\varphi^{q_2} = 0$$

$$\begin{aligned} \frac{\alpha_1}{1-\mu_1\alpha_1} &= \frac{(p_1-1)(q_2-1)-p_2q_1)((q_2+1)-q_1)}{(p_1-1)(q_2-1)-p_2q_1)-\mu_1(q_2+1)-q_1)}, \\ \frac{\alpha_2}{1-\mu_2\alpha_2} &= \frac{((p_1-1)(q_2-1)-p_2q_1)(p_2+1)-p_1]}{(p_1-1)(q_2-1)-p_2q_1)-\mu_2(p_2+1)-p_1)}, \end{aligned} \qquad (13)$$

$$\begin{aligned} L_3(f,\varphi) \quad \xi^{1-N} \frac{d}{d\xi} \; \xi^{N-1} f^{m_1-1} \left|\frac{df^k}{d\xi}\right|^{p-2} \frac{df}{d\xi} + \frac{\xi}{p}\frac{df}{d\xi} + a_1 \frac{(q_2+1)-q_1)}{(p_1-1)(q_2-1)-p_2 q_1)} f + \varepsilon f^{p_1}\varphi^{q_1} &= 0, \\ L_4(f,\varphi) \quad \xi^{1-N} \frac{d}{d\xi} \; \xi^{N-1} \varphi^{m_2-1} \left|\frac{d\varphi^k}{d\xi}\right|^{p-2} \frac{d\varphi}{d\xi} + \frac{\xi}{p}\frac{d\varphi}{d\xi} + a_2 \frac{(p_2+1)-p_1}{(p_1-1)(q_2-1)-p_2 q_1)} \varphi + \varepsilon f^{p_2}\varphi^{q_2} &= 0, \end{aligned} \qquad (14)$$

where

$$\begin{aligned} a_1 &= \frac{\left((p_1-1)(q_2-1)-p_2q_1\right)\left((p_2+1)-p_1\right)}{\left((p_1-1)(q_2-1)-p_2q_1\right)-\mu_1\left((p_2+1)-p_1\right)}, \\ a_2 &= \frac{\left((p_1-1)(q_2-1)-p_2q_1\right)\left((q_2+1)-q_1\right)}{\left((p_1-1)(q_2-1)-p_2q_1)\right)-\mu_2\left((q_2+1)-q_1\right)}, \end{aligned}$$

$$\mu_1 = k(p-2) + m_1 - 1, \;\; \mu_2 = k(p-2) + m_2 - 1$$

if

$$\mu_1\left((q_2+1)-q_1\right)=\mu_2\left(\left(p_1+1\right)-p_2\right),$$
$$\alpha_1=\frac{(q_2+1)-q_1}{\left(p_1-1\right)\left(q_2-1\right)-p_2q_1},\quad \alpha_2=\frac{\left(p_1+1\right)-p_2}{\left(p_1-1\right)\left(q_2-1\right)-p_2q_1}, \qquad (14)$$

Theorem11.1. Assume $\frac{\alpha_1}{1-\mu_1\alpha_1}<\frac{N}{p},\frac{\alpha_2}{1-\mu_1\alpha_2}<\frac{N}{p}$, $\mu_i>0,\ i=1,2\ ,u_0\left(x\right)\quad u_+\left(0,x\right),\ v_0\left(x\right)\quad v_+\left(0,x\right),\ x\quad R.$ Then for the solution of the problem (1)-(2) the following estimate $u\left(t,x\right)\quad u_+\left(\mathrm{t},x\right),\ v\left(t,x\right)\quad v_+\left(\mathrm{t},x\right)$ in $Q=\{(t,x):\ t>0,\ x\quad R^N\}$, holds.

Corollary. For the front perturbation we have the estimate

$$|x|<a^{\frac{p-1}{p}}\tau(\mathrm{t})^{1/p}=a^{\frac{p-1}{p}}\quad\frac{\left(\mathrm{T+t}\right)^{1\text{-}\alpha_1\mu_1}}{1\text{-}\alpha_1\mu_1}^{\frac{1}{p}}$$

The properties of finite speed of perturbation, a space localization of mathematic model of nonlinear processes heat conductivity, a diffusion, a filtration in liquid and gas in two componential medium with source or absorption based on self-similar analyses established. The condition of the Fujita type global solvability proved. Main problem choosing of appropriate initial approximation keeping properties of nonlinear system in numerical analysis of solution is solved. It should be noted that for numerical calculations of a nonlinear problem, it is very important to choose a suitable initial approximation, since this ensure convergence with a given accuracy to the solution of the problem with a minimum number of iterations. Result of the numerical solution showed effectivity suggested approach. For numerical solution the A.A., Samarskiy I.M. Sobol [4] type compositional difference scheme suggested.

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